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A short introduction to arbitrage theory and pricing in mathematical finance for discrete-time markets with or without frictions

Emmanuel Lepinette

Abstract

In these notes, we introduce the theory of arbitrage and pricing for frictionless models, i.e. the classical theory of mathematical finance. The main classical results are presented, namely the characterization of absence of arbitrage opportunities, based on convex duality. Dual characterizations of super-hedging prices are deduced. We then introduce financial market models with proportional transaction costs. We discuss no arbitrage conditions and characterize super-hedging prices as in the frictionless case. An alternative approach based on the liquidation value concept is finally presented.

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1 Markets without frictions

1.1 Introduction

We consider a discrete-time stochastic basis

$$(\Omega, (\mathcal{F}_t)_{t=0,1,\cdots,T}, P).$$

The set Ω is the space of all possible states of the financial market we consider on the period $[0,T]$. A state $\omega \in \Omega$ may be complicated; it may include risky asset prices but also preferences of agents acting on the market. For every t, we suppose that $\mathcal{F}_t$ is a σ-algebra, which is supposed to be complete, i.e. contains the negligible sets for the probability measure P. Recall that, by definition, the elements of $\mathcal{F}_t$ are subsets of Ω and we have the following properties:

(i) $\Omega, \emptyset \in \mathcal{F}_t$,

(ii) $F_t \in \mathcal{F}_t$ implies that $F_t^c := \Omega \setminus F_t \in \mathcal{F}_t$,

(iii) For all countable family $(F_t^n)_{n\geq 1}$ of $\mathcal{F}_t$, $\bigcup_n F_t^n$, $\bigcap_n F_t^n \in \mathcal{F}_t$.

Notice that $(\mathcal{F}_t)_{t=0,1,\cdots,T}$ is called a filtration in the sense that, for all $t < u$, $\mathcal{F}_t \subseteq \mathcal{F}_u$. The σ-algebra $\mathcal{F}_t$ models the information available at time t.

Example 1.1. Let us consider a financial market composed of d exchangeable assets whose prices are given at time t by the vector $S_t = (S_t^1, \cdots, S_t^d)$. We define

$$\mathcal{F}_t = \sigma\left(S_u : \ u \leq t\right), \quad t \geq 0,$$

as the smallest σ-algebra making the mappings $S_u : \omega \mapsto S_u(\omega)$, $u \leq t$, measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$. Such a σ-algebra exists as an intersection of any family of σ-algebras is a σ-algebra. We may verify that $(\mathcal{F}_t)_{t\geq 0}$ is a filtration.

In finance, we generally suppose that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is complete, i.e. $\mathcal{F}_0$ contains the negligible sets for P. Actually, the classical case is to consider $\mathcal{F}_0$ as the smallest σ-algebra containing the negligible sets. We may show that X_0 is $\mathcal{F}_0$-measurable if and only if there exists a constant c such that $P(X = c) = 1$, i.e. $X = c$ a.s. (almost surely).

The family of random variables $(X_t)_{t\geq 0}$ is said to be a stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if, for all $t \geq 0$, $X_t : \Omega \to \mathbb{R}^d$ is $\mathcal{F}_t$-measurable. This means that, for all B in the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, $X_t^{-1}(B) \in \mathcal{F}_t$. Notice that, if $\mathcal{F}_t$ is the information available at time t on the market, the $\mathcal{F}_t$-measurability means that X_t is observed at time t.

In the following, we describe a portfolio process by the quantities held by an agent acting on the market.

Precisely, a strategy $\hat{\theta} = (\theta^0, \theta)$ is such that θ^0_t is the quantity at time t invested in some non risky asset whose price is S^0_t at time t while $\theta_t = (\theta^1_t, \cdots, \theta^d_t)$ is the vector of quantities θ^i_t invested in risky asset number $i = 1, \cdots, d$ whose price is S^i_t at time t.

The asset S^0 is said non risky at time t if $var(S^0_t) = 0$, i.e. there is no uncertainty about the future prices S^0_t: out of a negligible set N, we have $S^0_t(\omega) = S^0_t$, for all $\omega \in N^c$. In particular, we know by advance the prices $(S^0_t)_{t\in[0,T]}$. A classical modeling of S^0 is given by the deterministic dynamics

$$\frac{S^0_{t+1} - S^0_t}{S^0_t} = r,$$

where the interest rate r is a constant and S^0_0 is given.

On the contrary, we say that the asset $(S_t)_{t\in[0,T]}$ is risky at time t if $var(S_t) > 0$. This means that the mapping $\omega \mapsto S_t(\omega)$ is not constant. Therefore, we do not know by advance the future values of $S_t(\omega)$ as it depends on the market state $\omega \in \Omega$. A classical example is to suppose that

$$\Delta S_{t+1} := S_{t+1} - S_t = \mu S_t + \sigma S_t G_{t+1}$$

where $(G_t)_{t=1,\cdots,T}$ is a family of i.i.d. random variables with common distribution $\mathcal{N}(0,1)$ and σ, μ are two constants. This means that the returns are normally distributed. Notice that when $\sigma = 0$, S is deterministic, i.e. is not risky.

Remark 1.2. There exists a continuous version of the model. The non risky asset satisfies the continuous time dynamics

$$dS^0_t = rS^0_t dt, \quad S^0_0 = 1.$$

The solution is given by $S^0_t = e^{rt}$ as it is the solution of the o.d.e. $(S^0_t)' = \frac{dS^0_t}{dt} = rS^0_t$. Notice that

$$r = \lim_{dt\to 0} \left(\frac{S^0_{t+dt} - S^0_t}{S^0_t} \right) / dt.$$

This means that r is interpreted as an instantaneous interest rate.

The risky asset is given by the Black and Scholes model, i.e. the price S follows the dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 \text{ is given.}$$

The stochastic process W is supposed to be a (standard) Brownian motion, i.e. W satisfies the following conditions:

1 For all t, W_t is $\mathcal{F}_t$-measurable and $W_0 = 0$.

2 With probability 1, the trajectories $t \mapsto W_t(\omega)$, $\omega \in \Omega$, are continuous.

3 For all $u < t$, $W_t - W_u$ is independent of $\mathcal{F}_u$.

4 If $t_4 - t_3 = t_2 - t_1$, then $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are equally distributed as $\mathcal{N}(0, t_4 - t_3)$ [1].

Let us interpret the dynamics of $(S_t)_{t\in[0,T]}$. We introduce the discrete dates $t^n_i = \frac{T}{n} i$, $i = 0, 1, \cdots, n$. We have $\Delta t^n_i := t^n_i - t^n_{i-1} = T/n$. As $n \to \infty$,

$$\Delta S_{t^n_{i+1}} := S_{t^n_{i+1}} - S_{t^n_i} \simeq \mu S_{t^n_i} \Delta t^n_i + \sigma S_{t^n_i} \Delta W_{t^n_{i+1}}, \quad i \geq 1,$$

where $\Delta W_{t_i} = \sqrt{T/n}\, G_i$ with $(G_i)_{i=1,\cdots,n}$ a family of i.i.d. random variables with common distribution $\mathcal{N}(0,1)$. This property is directly deduced from the definition of W. Notice that, when $\sigma = 0$, $S_t = S_0 e^{\mu t}$ is deterministic, i.e. it is non risky. If $\sigma > 0$, the Black and Scholes model supposes that the log-returns $\log(S_{t^n_{i+1}}/S_{t^n_i})$ are normally distributed. Indeed, we may show that $S_{t^n_{i+1}} = S_{t^n_i} e^{\sigma \Delta W_{t^n_{i+1}} + (\mu - \sigma^2/2)\Delta t^n_{i+1}}$. The coefficient σ is called the volatility. The larger σ is, the further could be S_t from the deterministic trajectory $S_0 e^{\mu t}$. Actually, we may show that $E(S_t) = S_0 e^{\mu t}$, $t \geq 0$.

For readers interested in stochastic calculus, very good notes by Jeanblanc M. are available in french [13] but also by Lamberton D. and Lapeyre B. in english [19].

1.2 Financial market without frictions

A financial market is said without frictions if there is no transaction costs when selling or buying risky assets. For a strategy $\hat{\theta} = (\theta^0, \theta)$, we define the liquidation value at time t:

$$V_t = V^{\hat{\theta}}_t = \theta^0_t S^0_t + \theta_t \cdot S_t = \theta^0_t S^0_t + \sum_{i=1}^d \theta^i_t S^i_t.$$

The stochastic process $V = V^{\hat{\theta}}$ is called the portfolio process associated to $\hat{\theta}$. Here, θ^i_i is allowed to be non positive (short position), which corresponds to a debt in the asset number i. The formulation above supposes that there is no transaction costs. Indeed, otherwise, when selling or buying risky assets to liquidate the positions given by $\hat{\theta}$, there should be a cost $c_t > 0$ to withdraw from the liquidation value.

In the following, we denote by $L^0(\mathbb{R}^n, \mathcal{F}_t)$, $n \geq 1$, the set of all $\mathcal{F}_t$-measurable random variables with values in $\mathbb{R}^n$.

Definition 1.3. An European option is a contract between two agents (seller and buyer) allowing the option holder (buyer) to get a terminal wealth ξ_T (called the payoff) from the seller at some fixed maturity $T > 0$. Such a contract is sold at time $t = 0$ at some price. The classical example is the so-called Call option, i.e. such that $\xi_T = (S_T - K)^+$ where $K \geq 0$ is a constant, which is called the strike. This means that, if $S_T \geq K$,

[1] It is a Gaussian distribution with mean 0 and variance $t_4 - t_3$

the option holder get $S_T - K$ and 0 otherwise. Such a contract corresponds to the possibility for the holder to buy the underlying asset S at price K at time T instead of the real price S_T. This is clearly interesting only if $S_T \geq K$, in which case the gain of the transaction is $S_T - K \geq 0$.

A fundamental problem in mathematical finance is to determine a price for an European option. To do so, we introduce the following definitions.

Definition 1.4. A portfolio process $V_t = V_t^{\hat{\theta}}$, $t = 0, \cdots, T$, is said self-financing if

$$\theta_{t-1}^0 S_t^0 + \theta_{t-1} \cdot S_t = \theta_t^0 S_t^0 + \theta_t \cdot S_t, \quad t = 1, \cdots, T.$$

For any stochastic process X, we introduce the following notations: $\Delta X_t = X_t - X_{t-1}$ for $t \geq 1$ and the discounted value $\tilde{X}_t = X_t / S_t^0$. We may easily show the following:

Lemma 1.5. *A portfolio process $V_t = V_t^{\hat{\theta}}$, $t = 0, \cdots, T$, is said self-financing if and only if $\Delta V_t = \theta_{t-1}^0 \Delta S_t^0 + \theta_{t-1} \cdot \Delta S_t$, $t = 1, \cdots, T$.*

Lemma 1.6. *A portfolio process $V_t = V_t^{\hat{\theta}}$, $t = 0, \cdots, T$, is said self-financing if and only if $\Delta \tilde{V}_t = \theta_{t-1} \cdot \Delta \tilde{S}_t$, $t = 1, \cdots, T$.*

In the following, we denote by $\mathcal{R}_0^T$ the set of all discounted terminal values $\tilde{V}_T$ of self-financing portfolio processes starting from the initial value $\tilde{V}_0 = V_0 = 0$. Writing $\tilde{V}_T = \tilde{V}_0 + \sum_{t=1}^T \Delta V_t$, we have

$$\mathcal{R}_0^T = \left\{ \sum_{t=1}^T \theta_{t-1} \cdot \Delta \tilde{S}_t \ , \ \theta_t \in L^0(\mathbb{R}, \mathcal{F}_t), t = 0, \cdots, T-1 \right\}.$$

We also introduce the set of super-hedgeable claims $\mathcal{A}_0^T = \mathcal{R}_0^T - L^0(\mathbb{R}_+, \mathcal{F}_T)$ we may obtain from a zero initial endowment, i.e. $\xi_T \in \mathcal{A}_0^T$ if and only if there exists $V_T \in \mathcal{R}_0^T$ such that $V_T \geq \xi_T$ a.s. In that case, we say that V super-replicates (or super-hedges) ξ_T at time T. Moreover, if $V_T = \xi_T$ a.s. we say that V replicates ξ_T.

Definition 1.7. A price for the payoff ξ_T is any initial value V_0 of a self-financing portfolio process V such that $V_T \geq \xi_T$ a.s. We denote by $\mathcal{P}(\xi_T)$ the set of all prices for ξ_T.

1.3 One step financial market: $T = 1$ and $d = 1$

In this section, we consider the simplest case where $T = 1$ and $d = 1$. It suffices to understand the main ideas in that case to extend them to the general case, up to some technical difficulties. Here, observe that we have $\mathcal{R}_0^T = \left\{ \theta_0 \cdot \Delta \tilde{S}_1 : \theta_0 \in \mathbb{R} \right\}$. A price for the payoff $\xi_1 \in L^0(\mathbb{R}, \mathcal{F}_1)$ is a value p_0 such that

$$\tilde{V}_1 = p_0 + \theta_0 \cdot \Delta \tilde{S}_1 \geq \tilde{\xi}_1 \text{ a.s. for some } \theta_0 \in \mathbb{R}.$$

This is equivalent to say that $\tilde{\xi}_1 - p_0 \in \mathcal{A}_0^1$. Therefore, the question is whether $\tilde{\xi}_1 - p_0 \in \mathcal{A}_0^1$ or not: $\tilde{\xi}_1 - p_0 \notin \mathcal{A}_0^1$. The last condition may be related to a convex separation problem as $\mathcal{A}_0^1$ is actually a convex cone. This is why, we prefer for $\mathcal{A}_0^1$ to be closed. The natural problem is to find a condition under which this is the case.

Notice that, if $d = 1$, S_0 is a price for $\xi_1 = (S_1 - K)^+$, $K \geq 0$. Indeed, it suffices to follow the buy and hold strategy $\theta_0 = (0, 1)$, i.e. buying one unit of the risky asset at price S_0. At $T = 1$, we obtain $\tilde{V}_1 = S_0 + \theta_0 \Delta \tilde{S}_1 = S_0 + (\tilde{S}_1 - S_0) = \tilde{S}_1$. So, $V_1 = S_1 \geq (S_1 - K)^+ = \xi_1$.

It is traditional to suppose the closedness of $\mathcal{A}_0^T$ to characterize the super-hedging prices. Of course, we need to precise the topology we use. In particular, $L^0(\mathbb{R}^d, \mathcal{F}_T)$ is endowed with the topology of the convergence in probability, so that it is a metric space: $d_0(X, Y) = E(|X - Y| \wedge 1)$. The spaces $L^p(\mathbb{R}^d, \mathcal{F}_T)$, $p \in [1, \infty]$ are endowed with the usual norms $\|X\|_p := (E|X|^p)^{1/p}$ if $p < \infty$ and $\|X\|_\infty$ is the usual norm for bounded random variables X of L^∞. With $\frac{1}{p} + \frac{1}{q}$, the topological dual of $L^p(\mathbb{R}^d, \mathcal{F}_T)$ is $L^q(\mathbb{R}^d, \mathcal{F}_T)$ for $p \leq 1$ but the dual of $L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is larger than $L^1(\mathbb{R}^d, \mathcal{F}_T)$ except if we endow $L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ with the $\sigma(L^\infty, L^1)$ topology.

Closedness of $\mathcal{A}_0^1$ in $L^0(\mathbb{R}, \mathcal{F}_1)$ for $d = 1$.

We denote by S the single risky asset. Let $X^n = \theta_0^n \Delta S_1 - \epsilon_n^+ \in \mathcal{A}_0^1$ where $(\theta_0^n)_{n \geq 1}$ is a sequence of $\mathbb{R}$ and $(\epsilon_n^+)_{n \geq 1}$ is a sequence in $L^0(\mathbb{R}_+, \mathcal{F}_1)$. Suppose that $X^n \to X$ a.s.

1st case: $\sup_n |\theta_0^n| < \infty$. In that case, the sequence $(\theta_0^n)_{n \geq 1}$ belongs to a compact set and there exists a subsequence such that that $\theta_0^n \to \theta_0 \in \mathbb{R}$. Therefore, $(\epsilon_n^+)_{n \geq 1}$ is almost surely convergent to some $\epsilon^+ \in L^0(\mathbb{R}_+, \mathcal{F}_1)$. We conclude that $X = \theta_0 \Delta S_1 - \epsilon^+ \in \mathcal{A}_0^1$.

2nd case: $\sup_n |\theta_0^n| = \infty$. In that case, we may suppose that $|\theta_0^n| \to \infty$. Let us define $\bar{\theta_0}^n = \theta_0^n / (1 + |\theta_0^n|)$. We define similarly $\bar{X}^n$ and $\bar{\epsilon}_n^+$. We have $\bar{X}^n = \bar{\theta_0}^n \Delta S_1 - \bar{\epsilon}_n^+ \in \mathcal{A}_0^1$ where $|\bar{\theta_0}^n| \leq 1$. Therefore, we may apply the first case and, as $n \to \infty$, we deduce an equality of the type $0 = \bar{\theta}_0 \Delta \tilde{S}_1 - \bar{\epsilon}^+$ where $|\bar{\theta}_0| = 1$. We deduce that $\bar{\theta}_0 \Delta S_1 \geq 0$ a.s.

As $d = 1$, $\bar{\theta}_0 = \pm 1$. Consider the case where $\bar{\theta}_0 = 1$, we have $\Delta S_1 \geq 0$ a.s. Otherwise, we have $\Delta S_1 \leq 0$ a.s. At this stage, we can not conclude anything about the closedness. Let us consider the case $\Delta S_1 \geq 0$ a.s. Consider the strategy $\hat{\theta}_0^n = n(-S_0, 1)$. Then, starting from the zero initial endowement, we get the terminal portfolio process $\tilde{V}_1^n = 0 + \theta_0^n \Delta S_1 = n \Delta S_1 \geq 0$ a.s. This means that from nothing (zero initial capital), we get a non negative terminal wealth, i.e. we do not take any risk to face a loss. Moreover, if n is large enough and if $P(\Delta S_1 > 0)$ there is a non null probability to get a strictly positive gain $n\Delta S_1 > 0$ as large as we want, i.e. we get what we call an arbitrage opportunity. If the agents acting on this financial market are well informed and rational, we may think that they all utilize this possibility to get positive money without taking any risk. Therefore, they all buy the risky asset to hold a position of type θ_0^n. Then, the risky asset price S_0 should go up and the condition $\Delta S_1 \geq 0$ a.s. should fail. This leads to the absence of arbitrage opportunity condition we now define.

Definition 1.8. An arbitrage opportunity is a terminal portfolio process $\tilde{V}_T \in \mathcal{R}_0^T$ starting from the zero initial endowment such that $\tilde{V}_T \geq 0$ a.s. and $P(\tilde{V}_T > 0) > 0$.

Definition 1.9. We say that the NA condition (No Arbitrage opportunity) holds if there is no arbitrage opportunity, i.e. $\mathcal{R}_0^T \cap L^0(\mathbb{R}_+, \mathcal{F}_T) = \{0\}$ or, equivalently, $\mathcal{A}_0^T \cap L^0(\mathbb{R}_+, \mathcal{F}_T) = \{0\}$.

Proposition 1.10. *If NA holds, then $\mathcal{A}_0^1$ is closed in $L^0(\mathbb{R}_+, \mathcal{F}_T)$.*

Proof. From above, it suffices to study the case where $\Delta\tilde{S}_1 \geq 0$ a.s. or $\Delta\tilde{S}_1 \leq 0$. In that case, $\pm\Delta\tilde{S}_1 \in \mathcal{A}_0^1 \cap L^0(\mathbb{R}_+, \mathcal{F}_1) = \{0\}$ hence $\Delta\tilde{S}_1 = 0$ a.s. hence $X^n = -\epsilon_n^+ \to X \leq 0$ a.s. This implies that $X \in -L^0(\mathbb{R}_+, \mathcal{F}_1) \subseteq \mathcal{A}_0^1$. The conclusion follows. □

The following step is to characterize the NA condition. To do so, we first recall a lemma, see [15, Lemma 2.1.3, Section 2.1.2]. This result may be seen as a generalization of the Halmos-Savage theorem, see e.g. [17].

Lemma 1.11. *Let $G = (\Gamma_i)_{i\in I}$ be a family of elements of a σ-algebra $\mathcal{F}$ such that, for all $\Gamma \in \mathcal{F}$, if $P(\Gamma) > 0$, there exists $i \in I$ such that $P(\Gamma \cap \Gamma_i) > 0$. Then, there exists a countable family $(\Gamma_{i_n})_{n\geq 1}$ such that $\Omega = \bigcup_{n=1}^{\infty} \Gamma_{i_n}$ a.s.*

Proof. We may suppose that G is stable under countable union. Indeed, in the contrary case, it suffices to replace G by $\tilde{G}$, which is the set of all countable unions of elements of G. Let us consider $m = \sup_i P(\Gamma_i)$. For a countable sequence, we have $m = \lim \uparrow P(\Gamma_{i_n}) = P(\hat{\Gamma})$ where $\hat{\Gamma} = \bigcup_{n=1}^{\infty} \Gamma_{i_n}$. We claim that $P(\hat{\Gamma}) = 1$, which is enough to conclude. Suppose by contradiction that $P(\hat{\Gamma}) < 1$ hence $P(\hat{\Gamma}^c) > 0$. By assumption, there exits $i_0 \in I$ such that $P(\hat{\Gamma}^c \cap \Gamma_{i_0}) > 0$. Therefore, as G is stable under countable union,

$$m \geq P(\hat{\Gamma} \cup \Gamma_{i_0}) = P(\hat{\Gamma}) + P(\hat{\Gamma}^c \cap \Gamma_{i_0}) > P(\hat{\Gamma}) = m.$$

We get a contradiction so we may conclude. □

Theorem 1.12. *Suppose that $T = 1 = d$. Then, NA holds if and only if there exists $Q \sim P$ such that $dQ/dP \in L^\infty(\mathbb{R}, \mathcal{F}_1)$ and $E_Q\tilde{S}_1 = S_0$.*

Proof. Before presenting the proof, let us recall that the probability measures Q and P are equivalent ($Q \sim P$) means that they admit the same negligible sets. By the Radon-Nikodym theorem, if $Q \sim P$, there exists $\rho \in L^1((0,\infty), P, \mathcal{F}_1)$ such that $dQ/dP = \rho$, i.e. $Q(A) = E_P(\rho 1_A)$ for all $A \in \mathcal{F}_1$. In particular, we have $E_Q(X) = E_P(\rho X)$ for all $X \in L^1(\mathbb{R}, \mathcal{F}_1, P)$.

Suppose that NA holds. The property still holds under $P' \sim P$. In particular, with $dP'/dP = \alpha e^{-|\tilde{S}_1|}$, we may suppose w.l.o.g. that $\tilde{S}_1$ is integrable under P. We know that $\mathcal{A}_0^1$ is a convex cone closed in $L^0(\mathbb{R}, \mathcal{F}_1, P)$ by Proposition 1.10. We deduce that $\mathcal{A}_0^1 \cap L^1(\mathbb{R}, \mathcal{F}_1, P)$ is also closed in $L^1(\mathbb{R}, \mathcal{F}_1, P)$ since the convergence in L^1 implies the convergence in probability. By NA, for all $x \in L^1(\mathbb{R}^+, \mathcal{F}_1) \setminus \{0\}$, $x \notin \mathcal{A}_0^1 \cap L^1(\mathbb{R}, \mathcal{F}_1)$. By the Hahn-Banach separation theorem, we deduce the existence of $\rho_x \in L^\infty(\mathbb{R}, \mathcal{F}_1)$ and $c \in \mathbb{R}$ such that

$$E(\rho_x X) < c < E(x\rho_x), \quad \forall X \in \mathcal{A}_0^1.$$

As $\mathcal{A}_0^1$ is a cone, replace X by kX and make $k \to \infty$. We get that $E(\rho_x X) \leq 0$ for all $X \in \mathcal{A}_0^1$. Since, $-L^0(\mathbb{R}_+, \mathcal{F}_1) \subseteq \mathcal{A}_0^1$, we then deduce that $\rho_x \geq 0$ a.s. With $X = 0$, we get that $c > 0$ and, as $\mathcal{R}_0^1$ is a vector space, $E(\rho_x X) = 0$ for all $X \in \mathcal{R}_0^1$. As $P(\rho_x > 0) > 0$ (see the strict inequality above), we may renormalize and suppose that $\|\rho_x\|_\infty = 1$.

Let us consider the family $G = (\Gamma_x)_{x\in I}$ where $I = L^1(\mathbb{R}^+, \mathcal{F}_1) \setminus \{0\}$ and $\Gamma_x = \{\rho_x > 0\}$. For any $\Gamma \in \mathcal{F}_1$ such that $P(\Gamma) > 0$, $x = 1_\Gamma \in I$. Therefore, $E(\rho_x 1_\Gamma) > 0$ hence $P(\Gamma_x \cap \Gamma) > 0$. By Lemma 1.11, we may write $\Omega = \bigcup_{i=1}^{\infty} \Gamma_{x_i}$. Let us define $\rho = \sum_{i=1}^{\infty} 2^{-i}\rho_{x_i}$. We have $\rho > 0$ a.s. and we may renormalize ρ such that $\rho \in L^\infty(\mathbb{R}^+, \mathcal{F}_1)$ and $E_P(\rho) = 1$. We then define $Q \sim P$ such that $dQ/dP = \rho$. We still have $E(\rho X) = 0$ for all $X \in \mathcal{R}_0^1$, in particular with $X = \Delta\tilde{S}_1 \in \mathcal{R}_0^1$, we may conclude that $E_Q(\tilde{S}_1) = S_0$.

Reciprocally, suppose the existence of $Q \sim P$ such that $E_Q(\tilde{S}_1) = S_0$. Take $\tilde{V}_T = \theta_0 \Delta\tilde{S}_1 \in \mathcal{R}_0^T \cap L^1(\mathbb{R}_+, \mathcal{F}_1)$. We have $E_Q(\tilde{V}_T) = E_Q(\theta_0\Delta\tilde{S}_1) = \theta_0 E_Q(\Delta\tilde{S}_1) = 0$. As $\tilde{V}_T \geq 0$ a.s., we get that $\tilde{V}_T = 0$, i.e. NA holds. □

A probability Q as in Theorem 1.12 is called a risk-neutral probability measure or (equivalent) martingale measure. Recall that, if $\xi_T \in L^1(\mathbb{R}, \mathcal{F}_1)$ is a payoff, a (super-replicating) price for ξ_T is an initial endowment p_0 of a portfolio process V satisfying $V_T \geq \xi_T$ a.s. We say that V replicates ξ_T when $V_T = \xi_T$ a.s. We denote by $\Gamma(\xi_T)$ the set of all prices for ξ_T.

Theorem 1.13. *Suppose that $T = 1$ and NA holds. Let us consider the set EMM$\neq \emptyset$ of all equivalent martingale measure. Then, if $\xi_T \in L^1(\mathbb{R}, \mathcal{F}_1)$,*

$$\Gamma(\xi_T) = \left[\sup_{Q\in EMM} E_Q(\tilde{\xi}_T), \infty\right).$$

Proof. Consider $p_0 \in \Gamma(\xi_T)$, i.e. there exists $\theta_0 \in \mathbb{R}$ such that $p_0 + \theta_0 \Delta \tilde{S}_1 \geq \tilde{\xi}_T$ a.s. Taking the Q-expectation, we get $p_0 \geq E_Q(\tilde{\xi}_T)$ since $E_Q(\theta_0 \Delta \tilde{S}_1) = \theta_0 E_Q(\Delta \tilde{S}_1) = 0$. It remains to show that

$$p_0^* = \sup_{Q \in EMM} E_Q(\tilde{\xi}_T) \in \Gamma(\xi_T).$$

Let us suppose by contradiction that $p_0^* \notin \Gamma(\xi_T)$, i.e. $\tilde{\xi}_T - p_0^* \notin \mathcal{A}_0^1$. As the latter set is a closed convex set in L^1, the Hahn-Banach separation theorem applies and we get $Z \in L^\infty(\mathbb{R}^d, \mathcal{F}_1)$, $c > 0$, such that

$$E(ZX) < c < E(Z(\tilde{\xi}_T - p_0^*)), \quad \forall X \in \mathcal{A}_0^1.$$

As in Theorem 1.12, we get that $Z \geq 0$ and $E(Z \Delta \tilde{S}_1) = 0$. Consider $Z_1 = dQ_1/dP$ where $Q_1 \in EMM \neq \emptyset$. We define $\rho = \alpha(\beta Z + Z_1)$ where $\alpha, \beta > 0$. We have

$$E(\rho(\tilde{\xi}_T - p_0^*)) = \alpha\left(\beta E(Z(\tilde{\xi}_T - p_0^*)) + E(Z_1(\tilde{\xi}_T - p_0^*))\right).$$

As $E(Z(\tilde{\xi}_T - p_0^*)) > 0$, we may choose $\beta > 0$ large enough in such a way that $E(\rho(\tilde{\xi}_T - p_0^*)) > 0$. We then fix α such that $\rho > 0$ defines an equivalent probability measure $Q \sim P$ with $dQ/dP = \rho$. Moreover, by construction, $E_Q(\tilde{S}_1) = 0$, i.e. $Q \in EMM$. It follows that $p_0^* \geq E_Q(\tilde{\xi}_T)$. On the other hand, $E_Q(\tilde{\xi}_T) > p_0^*$ by construction hence a contradiction. □

A natural question is whether EMM is a singleton. This is related to the concept of completeness for the market.

Definition 1.14. We say that the financial market is complete if for any $\xi_T \in L^1(\mathbb{R}, \mathcal{F}_T)$, there exists a self-financing portfolio process V such that $V_T = \xi_T$.

Proposition 1.15. *Let $T = 1$. Suppose that NA holds. Then, the market is complete if and only if EMM is a singleton.*

Proof. Suppose that the market is complete. Let $Q_1, Q_2 \in EMM$. Consider $A \in \mathcal{F}_1$. The payoff $\xi_T = 1_A$ is replicable by assumption, i.e. there exists a self-financing portfolio process V such that $V_T = 1_A$. We have $\tilde{V}_T = V_0 + \theta_0 \Delta \tilde{S}_1$ for some $\theta_0 \in \mathbb{R}$, hence $E_{Q_1} \tilde{V}_T = E_{Q_2} \tilde{V}_T = V_0$. This implies that $Q_1(A) = Q_2(A)$, for all A, i.e. $Q_1 = Q_2$.

Reciprocally, if $EMM = \{Q\}$, we know by Theorem 1.13, that $p_0^* = E_Q(\tilde{\xi}_1)$ is a super-replication price, i.e. there exists a portfolio process V such that $\tilde{V}_1 \geq \tilde{\xi}_1$. This means that $\tilde{\xi}_1 = p_0^* + \theta_0 \tilde{S}_1 - \epsilon_1^+$ where $\epsilon_1^+ \in L^0(\mathbb{R}_+, \mathcal{F}_1)$. We deduce that $E_Q(\tilde{\xi}_1) = p_0^* - E_Q(\epsilon_1^+)$ hence $E_Q(\epsilon_1^+) = 0$ and $\epsilon_1^+ = 0$. This implies that $\tilde{\xi}_1$ is replicable. □

1.4 General case: the Dalang-Morton-Willinger theorem

In this section, we generalize the results of the last section.

Definition 1.16. Let $Q \sim P$. We say that the stochastic process $(M_t)_{t=0,\cdots,T}$ is a Q-martingale if, for all $t = 0, \cdots, T$, M_t is Q-integrable ($E_Q|M_t| < \infty$) and $E_Q(M_{t+1}|\mathcal{F}_t) = M_t$.

We shall need a generalized version of the conditional expectation which allows to consider conditional expectation of non integrable random variables. Recall that the conditional expectation of any non negative random variable X exists and is defined as $E(|X||\mathcal{G}) = \lim_n \uparrow E(|X| \wedge n|\mathcal{G})$ where $|X| \wedge n \in [0, n]$, $n \geq 1$, is integrable.

Definition 1.17. Let $\mathcal{G} \subseteq \mathcal{F}$ be two σ-algebras and $X \in L^0(\mathbb{R}^d, \mathcal{F})$, $d \geq 1$. We say that X admits a conditional expectation $E(X|\mathcal{G})$ if $E(|X||\mathcal{G}) < \infty$ a.s. In that case, we define

$$E(X|\mathcal{G}) = E(X^+|\mathcal{G}) - E(X^-|\mathcal{G}) \in L^0(\mathbb{R}^d, \mathcal{G}).$$

We may show the following:

Lemma 1.18. *Let $X_\mathcal{G} \in L^0(\mathbb{R}^d, \mathcal{G})$ and suppose that $Y \in L^0(\mathbb{R}^d, \mathcal{F})$ admits a conditional expectation $E(Y|\mathcal{G})$. Then, $X_\mathcal{G} Y$ admits a conditional expectation such that $E(X_\mathcal{G} Y|\mathcal{G}) = X_\mathcal{G} E(Y|\mathcal{G})$.*

Proposition 1.19. *Suppose that NA holds. Then, $\mathcal{A}_0^T$ is closed in L^0.*

Proof. We show the statement by induction. For two dates, let us consider $X^n = \theta_{T-1}^n \tilde{S}_T - \epsilon_T^{n+} \in \mathcal{A}_{T-1}^T$ converging a.s. to X as $n \to \infty$. We suppose that $\theta_{T-1}^n \in L^0(\mathbb{R}^d, \mathcal{F}_{T-1})$ and $\epsilon_T^{n+} \in L^0(R_+, \mathcal{F}_T)$. We split Ω into two subsets:

a) On the set $\Omega_{T-1} = \{\liminf_n |\theta_{T-1}^n| < \infty\} \in \mathcal{F}_{T-1}$. By [15, Lemma 2.1.2, Section 2.1.2], there exists a random sequence $n_k \in L^0(\mathbb{N}, \mathcal{F}_{T-1})$ such that $\theta_{T-1}^{n_k}$ converges almost surely to some θ_{T-1}. Notice that

$$\theta_{T-1}^{n_k} = \sum_{j=k}^{\infty} \theta_{T-1}^j 1_{n_k=j} \in L^0(\mathbb{R}^d, \mathcal{F}_{T-1}).$$

We deduce that $\theta_{T-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{T-1})$. At last, we get that $\epsilon_T^{n+} \to \epsilon_T^+ \in L^0(R_+, \mathcal{F}_T)$. Finally, $X 1_{\Omega_{T-1}} = \theta_{T-1} 1_{\Omega_{T-1}} \tilde{S}_T - \epsilon_T^+ 1_{\Omega_{T-1}} \in \mathcal{A}_{T-1}^T$.

b) On the set $\Omega_{T-1}^c = \{\liminf_n |\theta_{T-1}^n| = \infty\}$. We use the normalization procedure, as in the last section, of the type $\bar{X} = X/(1 + |\theta_{T-1}^n|)$. Then, we apply the first step a) to the sequence $\bar{X}^n$. In limit, we get that $\bar{\theta}_{T-1} \tilde{S}_T - \bar{\epsilon}_T^+ = 0$ for some $\bar{\epsilon}_T^+ \geq 0$ a.s. and $\bar{\theta}_{T-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{T-1})$ such that $|\bar{\theta}_{T-1}| = 1$. As $\bar{\theta}_{T-1} \tilde{S}_T = \bar{\epsilon}_T^+ \in \mathcal{A}_{T-1}^T \cap L^0(\mathbb{R}_+, \mathcal{F}_{T-1})$, we get that $\bar{\theta}_{T-1} \tilde{S}_T = 0$ by NA.

Let us restrict ourselves to the case $d = 1$. We shall see the general case below. As $\bar{\theta}_{T-1} \in \{-1, 1\}$, we get that $\Delta\tilde{S}_T = 0$ hence $X^n \leq 0$ and $X \leq 0$. Therefore, $X1_{\Omega^c_{T-1}} \in \mathcal{A}^T_{T-1}$. We conclude that $X = X1_{\Omega_{T-1}} + X1_{\Omega^c_{T-1}} \in \mathcal{A}^T_{T-1}$.

Suppose by induction that $\mathcal{A}^T_t$ is closed and let us show that $\mathcal{A}^T_{t-1}$ is also closed. To do so, consider a converging sequence

$$X^n = \theta^n_{t-1}\Delta\tilde{S}_t + \cdots + \theta^n_{T-1}\Delta\tilde{S}_T - \epsilon^{n+}_T \to X.$$

c) On the set $\Omega_{t-1} = \{\liminf_n |\theta^n_{t-1}| < \infty\} \in \mathcal{F}_{t-1}$, we may suppose w.l.o.g. (see the first step a)) that $\theta^n_{t-1} \to \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$. Therefore, $\theta^n_t\Delta\tilde{S}_{t+1} + \cdots + \theta^n_{T-1}\Delta\tilde{S}_T - \epsilon^{n+}_T$ is convergent by the induction hypothesis to $X_{t,T} \in \mathcal{A}^T_t$. It follows that $X1_{\Omega_{t-1}} = \theta_{t-1}1_{\Omega_{t-1}}\Delta\tilde{S}_t + X_{t,T}1_{\Omega_{t-1}} \in \mathcal{A}^T_{t-1}$.

d) On the set $\Omega^c_{t-1} = \{\liminf_n |\theta^n_{t-1}| = \infty\} \in \mathcal{F}_{t-1}$, we use the normalization procedure as in b) and we deduce an equality of the type

$$\gamma_{t-1} = \bar{\theta}_{t-1}\Delta\tilde{S}_t + \cdots + \bar{\theta}_{T-1}\Delta\tilde{S}_T - \bar{\epsilon}^+_T = 0.$$

In the case where $d = 1$, $\bar{\theta}_{t-1} \in \{-1, 1\}$. We split $\Omega^c_{t-1} = \{\bar{\theta}_{t-1} = -1\} \cup \{\bar{\theta}_{t-1} = 1\}$. On the set $\{\bar{\theta}_{t-1} = 1\}$, we observe that

$$\begin{aligned} X^n &= X^n - \theta^n_{t-1}\gamma_{t-1} \\ &= \theta^n_t\Delta\tilde{S}_{t+1} + \cdots + \theta^n_{T-1}\Delta\tilde{S}_T - \epsilon^{n+}_T \in \mathcal{A}^T_t. \end{aligned}$$

Using the induction hypothesis, we may conclude. Similar arguments apply on the set $\{\bar{\theta}_{t-1} = -1\}$.

The general case where $d > 1$ needs to be thought component-wise. As $|\bar{\theta}_{t-1}| = 1$, we split Ω^c_{t-1} into a partition $(B_i)_{i=1,\cdots,d}$ of $\mathcal{F}_{t-1}$ such that $B_i \subseteq \{\bar{\theta}^i_{t-1} \neq 0\}$. On each B_i, we assume w.l.o.g. that $\theta^{ni}_t \neq 0$ and we write $X^n = X^n - \alpha^n_{t-1}\gamma_{t-1}$ where α^n_{t-1} is chosen such that it is possible to rewrite X^n in such a way that $\theta^{ni}_{t-1} = 0$, i.e. we have strictly reduced the number of non null components of θ^n_{t-1}. We then go to step c) and, if necessary, we still reduce the number of non null components of θ^n_{t-1}. As d is finite, we may conclude as, in the worst case, θ^n_{t-1} is finally reduced to 0 so that the induction hypothesis applies. Another technique is to define almost surely a matrix P^n such that $P^n\bar{\theta}_{t-1} = \theta^n_t$ and observe that $X^n = X^n - P^n\gamma_{T-1}$. In that case, we need do show that P^n is $\mathcal{F}_{t-1}$-measurable. This may be proven by a measurable selection argument. □

The following result is fundamental in the theory of arbitrage theory. A complete version is given in [15]. We provide a proof which is not exactly the original one. Note that there are other available proofs, see [26], [24] and [23].

Theorem 1.20 (Dalang-Morton-Willinger theorem). *The condition NA holds if and only if there exists $Q \sim P$ such that $(\tilde{S}_t)_{t=0,\cdots,T}$ is a Q-martingale.*

Proof. Suppose that NA holds. We know by Proposition 1.19 that $\mathcal{A}^T_0$ is closed in L^0. So, we apply the reasonings we did for $T = 1$ and we deduce $Q \sim P$ such that $E_Q(X) = 0$ for all $X \in \mathcal{R}^T_0$. In particular, for all $t \geq 1$, for all $F_{t-1} \in \mathcal{F}_{t-1}$, $1_{F_{t-1}}\Delta\tilde{S}_t \in \mathcal{A}^T_0$ hence $E_Q(1_{F_{t-1}}\Delta\tilde{S}_t) = 0$. This implies that $E_Q(\Delta\tilde{S}_t|\mathcal{F}_{t-1}) = 0$, i.e. $\tilde{S}$ is a Q-martingale.

Reciprocally, suppose that $\tilde{S}$ is a Q-martingale. Consider $\tilde{V}_T \in \mathcal{A}^T_0 \cap L^0(\mathbb{R}_+, \mathcal{F}_T)$. We write $\tilde{V}_T = \tilde{V}_{T-1} + \theta_{T-1}\Delta\tilde{S}_T$ where $\theta_{T-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{T-1})$. As $\Delta\tilde{S}_T$ is Q-integrable, we deduce that $\tilde{V}_T$ admits a generalized conditional expectation such that $E_Q(\tilde{V}_T|\mathcal{F}_{T-1}) = \tilde{V}_{T-1}$. We repeat the argument and we get that $E_Q(\tilde{V}_T|\mathcal{F}_{T-2}) = \tilde{V}_{T-2}$. Finally, we have $E_Q(\tilde{V}_T) = \tilde{V}_0 = 0$. As $\tilde{V}_T \geq 0$ a.s., $\tilde{V}_T = 0$, i.e. NA holds. □

As in the case $T = 1$, we also deduce the following dual characterization of the prices from the set EMM of all equivalent martingale measures $Q \sim P$ under which $\tilde{S}$ is a Q-martingale.

Theorem 1.21. *Suppose that NA holds. Consider $\xi_T \in L^1(\mathbb{R}, \mathcal{F}_T)$. The set of all super-hedging prices of ξ_T is*

$$\Gamma(\xi_T) = \left[\sup_{Q\in EMM} E_Q(\tilde{\xi}_T), \infty\right).$$

We may also introduce the largest sub-hedging price p for ξ_T, i.e. the largest price p such that $p + V_T \leq \xi_T$ a.s. for some self-financing portfolio process V. By symmetry, we have $p = \inf_{Q\in EMM} E_Q(\tilde{\xi}_T)$. Now, consider an extended market model where the payoff ξ_T is quoted at price $p(\xi_T)$ at time 0 and is available only at time T at the price ξ_T. Therefore, a terminal discounted claim is of the form $\sum_{t=1}^T \theta_{t-1}\Delta\tilde{S}_t + \theta'(\tilde{\xi}_T - p(\xi_T))$. Suppose that there exists an arbitrage opportunity for this extended market. In particular, for some strategy (θ, θ'), $\sum_{t=1}^T \theta_{t-1}\Delta\tilde{S}_t + \theta'_0(\tilde{\xi}_T - p(\xi_T)) \geq 0$ a.s. If $\theta'_0 = 0$, we get an arbitrage opportunity for the initial market contrarily to the assumption. If $\theta'_0 > 0$, divide by θ' and take the expectation for any $Q \in EMM$. We get that $p(\xi_T) \leq E_Q(\tilde{\xi}_T)$ hence $p(\xi_T) \leq \inf_{Q\in EMM} E_Q(\tilde{\xi}_T)$. Otherwise, if $\theta'_0 < 0$, we get that $p(\xi_T) \geq \sup_{Q\in EMM} E_Q(\tilde{\xi}_T)$. Therefore,

$$p(\xi_T) \in \left(\inf_{Q\in EMM} E_Q(\tilde{\xi}_T), \sup_{Q\in EMM} E_Q(\tilde{\xi}_T)\right)$$

implies that there is no arbitrage opportunity.

Conclusion: We have presented the main ideas of arbitrage theory without frictions and in discrete-time, i.e.

a no arbitrage condition NA is considered to ensure the closedness of the set of hedgeable claims. The NA condition is equivalent to the existence of a probability risk measure under which the discounted prices are martingales. At last, it is possible to dually characterize the super-hedging prices under NA via the dual elements, i.e. probability risk measures. For a deeper study of arbitrage theory for frictionless models, we send the readers to [15, Section 2].

2 Markets with frictions

2.1 Introduction

The theory we present in this section is rather recent. Most of the main results of the literature have been developed in the last fifteen years. A pioneering work is the paper by Jouini and Kallal [14] where bid and ask prices are considered. We propose in this section an introduction to financial market models with proportional transaction costs. In the following, we consider a discrete-time stochastic basis $(\Omega, (\mathcal{F}_t)_{t=0,1,\cdots,T}, P)$. We denote by e_1 the vector of $\mathbb{R}^d$, $d \geq 1$, such that the only non null component is the first one which is fixed to 1.

Example. Suppose that the market is composed of two assets. The first one is non risky and its (discounted) value is $S_t^0 = 1$ for all $t \in [0, T]$. The second asset is risky and the price is S_t at time t. As usual, we suppose that S is a stochastic process adapted to the filtration. We suppose that we need to pay proportional transaction costs when buying or selling the risky asset. Precisely, when buying one unit of the risky asset, we pay the price $S_t(1+\epsilon) = S_t + S_t\epsilon$. When selling one unit of the risky asset, we get the price $S_t(1-\epsilon) = S_t - S_t\epsilon$. This means that the proportional transaction cost rate is $\epsilon > 0$.

In this setting, we denote by V a portfolio process. Contrarily to the frictionless model, V is expressed in physical units, i.e. V is the strategy $\hat{\theta}$ of the last section. This choice is motivated by technical reasons: the dynamics of a portfolio process is not trivial with transaction costs. In the sequel, a financial position (x, y) describes the quantity $x \in \mathbb{R}$ and $y \in \mathbb{R}$ invested in assets S^0 and S respectively.

Definition 2.1. The liquidation value at time t of the financial position (x, y) is

$$L_t((x, y)) := x + y^+ S_t(1-\epsilon) - y^- S_t(1+\epsilon).$$

This definition is clear. If $y > 0$, we liquidate the long position by selling the y units of risky asset at price $S_t(1-\epsilon)$. If $y < 0$, we liquidate the short position by buying the y^- units of risky asset at price $S_t(1+\epsilon)$. Notice that L is linear only in the first component. This linearity is used afterwards.

Definition 2.2. At time t, the solvency set is defined as

$$G_t(\omega) := \{z = (x, y) \in \mathbb{R}^2 : \ L_t(z) \geq 0\}.$$

G_t is the set of all positions we may liquidate without any debt. Indeed, if $z \in G_t$, write $z = z - L_t(z)e_1 + L_t(z)e_1$ and observe that $L_t(z - L_t(z)e_1) = 0$. We may easily show that G_t is a closed convex cone. For $y \geq 0$, $z = (x, y) \in G_t$ if and only if $x + yS_t(1-\epsilon) \geq 0$, i.e. $zg_t^{2*} \geq 0$ where $g_t^{2*} = (1, S_t(1-\epsilon))$. For $y < 0$, $z = (x, y) \in G_t$ if and only if $x + yS_t(1+\epsilon) \geq 0$, i.e. $zg_t^{1*} \geq 0$ where $g_t^{1*} = (1, S_t(1+\epsilon))$. The vectors g_t^{1*} and g_t^{2*} are the generators of the positive dual cone

$$G_t^* = \{z \in \mathbb{R}^2 : \ zg_t \geq 0, \quad \forall g_t \in G_t\} = \text{cone}\,(g_t^{1*}, g_t^{2*}).$$

Lemma 2.3. *The solvency set is $\mathcal{F}_t$-graph-measurable at time t:*

$$\text{graph}\, G_t := \{(\omega, z) \in \Omega \times \mathbb{R}^d : \ z \in G_t(\omega)\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d).$$

Proof. It suffices to observe that $(\omega, z) \in \text{graph}\,(G_t)$ if and only if $zg_t^{1*} \geq 0$ and $zg_t^{2*} \geq 0$. □

Similarly, we have $G_t = \text{cone}\,(g_t^1, g_t^2)$, where $g_t^1 = (S_t(1+\epsilon), -1)$ and $g_t^2 = (-S_t(1-\epsilon), 1)$. Therefore, G_t^* is $\mathcal{F}_t$-graph-measurable at time t since $(G_t^*)^* = G_t$.

Proposition 2.4. *For all $z \in \mathbb{R}^2$,*

$$L_t(z) = \max\{\alpha \in \mathbb{R} : z - \alpha e_1 \in G_t\}.$$

Proof. Consider $\alpha \in \mathbb{R}$ such that $z - \alpha e_1 \in G_t$. Then, $L_t(z - \alpha e_1) \geq 0$, i.e. $L_t(z) - \alpha \geq 0$ hence $\alpha \leq L_t(z)$. Moreover, $L_t(z - L_t(z)e_1) = 0$ implies that $z - L_t(z)e_1 \in G_t$. The conclusion follows. □

Definition 2.5. A self-financing portfolio process is a stochastic process $(V_t)_{t=0,\cdots,T}$ starting from an initial endowment $V_{-1} = V_{0-}$ such that, for all $t \geq 0$, $\Delta V_t \in -G_t$ a.s.

The interpretation of the dynamics above is the following: we may write $V_{t-1} = V_t + (-\Delta V_t)$ so that it is possible to change V_{t-1} into V_t as it is allowed to let aside $(-\Delta V_t)$ whose liquidation value is non negative. Observe that the terminal value of V is an element of $V_{0-} + \sum_{t=0}^T L^0(-G_t, \mathcal{F}_t)$.

Several no arbitrage conditions have been considered in order to solve the super-hedging problem, as in the frictionless case. To do so, we consider the set of all terminal claims $\mathcal{A}_0^T$ we may obtain from a zero initial capital. We have

$$\mathcal{A}_0^T = \sum_{t=0}^T L^0(-G_t, \mathcal{F}_t).$$

We suppose that the prices are non negative. Therefore, $\mathbb{R}_+^2 \subseteq G_t$ a.s. hence $-L^0(\mathbb{R}_+^2, \mathcal{F}_T) \subseteq \mathcal{A}_0^T$.

Definition 2.6. NA^w: $\mathcal{A}_0^T \cap L^0(\mathbb{R}_+^2, \mathcal{F}_T) = \{0\}$.

Proposition 2.7. *Suppose that $S_T > 0$ a.s. and $\epsilon < 1$. The condition NA^w holds if and only if, for all $V_T \in \mathcal{A}_0^T$, $L_T(V_T) \geq 0$ implies that $L_T(V_T) = 0$ a.s.*

Proof. Suppose that NA^w holds and consider $V_T \in \mathcal{A}_0^T$ such that $L_T(V_T) \geq 0$. Since $V_T - L_T(V_T)e_1 \in G_T$ and G_T is stable under addition, we deduce that $L_T(V_T)e_1 = V_T - (V_T - L_T(V_T)e_1) \in \mathcal{A}_0^T \cap L^0(\mathbb{R}_+^2, \mathcal{F}_T) = \{0\}$, i.e. $L_T(V_T) = 0$ a.s. Reciprocally, consider $V_T \in \mathcal{A}_0^T \cap L^0(\mathbb{R}_+^2, \mathcal{F}_T)$. Necessarily, $L_T(V_T) \geq 0$ hence $L_T(V_T) = 0$. As $V_T \in \mathbb{R}_+^2$, we have $0 = L_T(V_T) = V_T^1 + V_T^2 S_T(1-\epsilon)$. It follows that $V_T^1 = V_T^2 = 0$. □

Clearly, the meaning of NA^w is the same than the NA condition of the frictionless case. In general, we shall see that stronger conditions are considered in presence of transaction costs to ensure the closedness of $\mathcal{A}_0^T$. In the following, we introduce the stochastic preorder $x \geq_{G_t} y$ if and only if $x - y \in G_t$, $t = 0, \cdots, T$.

Definition 2.8. An endowment for the payoff $\xi_T \in L^0(\mathbb{R}^2, \mathcal{F}_T)$ is a vector $p_0 \in \mathbb{R}^2$ which is the initial capital of a self-financing portfolio V such that $V_T \geq_{G_T} \xi_T$ a.s.

Notice that $p_0 \in \mathbb{R}^2$ is an endowment if $p_0 - \sum_{t=1}^T g_t = \xi_T + g_T'$ for some $g_t \in L^0(G_t, \mathcal{F}_t)$, $t \leq T$ and $g_T' \in L^0(G_T, \mathcal{F}_T)$. As G_T is a convex cone, we get that $p_0 + V_T = \xi_T$ where $V_T \in \mathcal{A}_0^T$. As in the frictionless case, let us see whether $\mathcal{A}_0^T$ may be closed. Let us start with $T = 1$.

Lemma 2.9. *Suppose that $T = 1$ and S_1 is not deterministic. Then, $\mathcal{A}_0^1$ is closed in probability under NA^w.*

Proof. Consider a convergent sequence $X^n = -g_0^n - g_1^n$ where $g_t^n \in -G_t$ a.s., $t = 0, 1$. We denote by X the limit of $(X^n)_{n\geq 1}$.

1) First case: $\sup_n |g_0^n| < \infty$. We may suppose that $g_0^n \to g_0 \in G_0$ as $(G_t)_{t=0,\cdots,T}$ take closed values. Therefore, $g_1^n \to g_1 \in L^0(G_1, \mathcal{F}_1)$ hence $X = -g_0 - g_1 \in \mathcal{A}_0^1$.

2) Second case: $\sup_n |g_0^n| = \infty$. We may suppose that $|g_0^n| \to \infty$. We normalize the sequence by setting $\bar{g}_0^n = g_0^n / |g_0^n|$, $\bar{X}^n = X^n / |g_0^n|$, etc. By the first case, we get an equality of the type $g_0 + g_1 = 0$ where $g_t \in G_t$ a.s., $t = 0, 1$. Therefore $-g_0 \in \mathcal{A}_0^1$ is such that $L_1(-g_0) = L_1(g_1) \geq 0$ hence $L_1(g_1) = L_1(-g_0) = 0$ by NA^w. Moreover, $g_1 = -g_0$ is deterministic. In the case where the second component g_1^2 of g_1 is non negative, $L_1(g_1) = 0$ means that $g_1^1 + g_1^2 S_1(1-\epsilon) = 0$. If $g_1^2 = 0$, then $g_1^1 = 0$ hence $g_0 = -g_1 = 0$ in contradiction with $|g_0| = 1$. So, $g_1^2 \neq 0$ and $S_1 = -g_1^1/(g_1^1(1-\epsilon))$. This leads to a contradiction. Similarly, the case where $g_1^2 \leq 0$ is excluded. □

Theorem 2.10. *Suppose that $T = 1$ and S_1 is not deterministic. Then, NA^w holds if and only if there exists a process $(Z_t)_{t=0,1}$ such that $Z_0 = E(Z_1)$ and $Z_t \in G_t^*$ a.s., $t = 0, 1$.*

Proof. Suppose that NA^w holds. Then, by Lemma 2.9, we deduce that $\mathcal{A}_0^1 \cap L^1(\mathbb{R}^2, \mathcal{F}_1)$ is closed in $L^1(\mathbb{R}^2, \mathcal{F}_1)$. Moreover, for any $x \in L^1(\mathbb{R}_+^2, \mathcal{F}_1) \setminus \{0\}$, $x \notin \mathcal{A}_0^1$ by NA^w. Therefore, by the Hahn-Banach separation theorem, there exists $Z_x \in L^\infty(\mathbb{R}^2, \mathcal{F}_1)$ and $c > 0$ such that $E(XZ_x) < c < E(xZ_x)$ for all $X \in \mathcal{A}_0^1$. Note that $Z_x \neq 0$ and we may assume that $\|Z_x\|_\infty = 1$. As $\mathcal{A}_0^1$ is a cone, we deduce that $E(XZ_x) \leq 0$ for all $X \in \mathcal{A}_0^1$. With $X = -g_0$ where g_0 is chosen arbitrarily in G_0, we deduce that $g_0 E(Z_x) \geq 0$ for any $g_0 \in G_0$, i.e. $E(Z_x) \in G_0^*$. Similarly, we have $E(Z_x g_1) \geq 0$ for all $g_1 \in L^1(G_1, \mathcal{F}_1)$. We deduce that $Z_x \in G_1^* \subseteq \mathbb{R}_+^2$ a.s. Indeed, otherwise, we may construct pointwise $g_1 \in L^0(G_1, \mathcal{F}_1)$ with $|g_1| = 1$ such that $Z_x g_1 \leq 0$ a.s. and $P(Z_x g_1 < 0) > 0$, i.e. a contradiction. To do so, we apply [15, Theorem 5.4.1, Section 5.4] which asserts that a $\mathcal{F}_1$-measurable selection g_1 exits as soon as the existence holds pointwise. We now consider the family $(G_x)_{x \in L^1(\mathbb{R}_+^2, \mathcal{F}_1)\setminus\{0\}}$ with $G_x := \{Z_x e_1 > 0\}$. For any Γ such that $P(\Gamma) > 0$, consider $x = 1_\Gamma e_1 \in L^1(\mathbb{R}_+^2, \mathcal{F}_1) \setminus \{0\}$. As $E(xZ_x) > 0$, we deduce that $P(\Gamma \cap \{Z_x e_1 > 0\}) > 0$. Therefore, Lemma 1.11 applies: we have $\Omega = \bigcup_{i=1}^\infty \{Z_x e_1 > 0\}$ for some countable family $(x_i)_{i \geq 1}$. We finally conclude with $Z_1 = \sum_{i \geq 1} 2^{-i} Z_{x_i}$ such that $Z_1 e_1 > 0$ and $Z_0 = E(Z_1)$.

Reciprocally, suppose the existence of Z and consider $V_1 = -g_0 - g_1 \in \mathcal{A}_0^1 \cap L^1(\mathbb{R}_+^2, \mathcal{F}_1)$. Then $Z_1 V_1 \geq 0$ and $Z_1 V_1 = 0$ if and only if $V_1 = 0$ as $Z_1 \in G_1^* \subseteq \operatorname{int} \mathbb{R}_+^2$. On the other hand, $E(Z_1 V_1) = -g_0 Z_0 - E(g_1 Z_1) \leq 0$ by assumption. Therefore, $Z_1 V_1 = 0$ and $V_1 = 0$. □

Definition 2.11. A consistent price system (CPS) is a P-martingale $(Z_t)_{t=0,\cdots,T}$ adapted to the filtration $(\mathcal{F}_t)_{t=0,\cdots,T}$ such that $Z_t \in G_t^* \setminus \{0\}$ a.s. for all $t = 0, \cdots, T$.

The following theorem (see [9]) is a generalization of Theorem 2.10 for $d = 2$, see [15, Theorem 3.2.15, Section 3].

Theorem 2.12 (Grigoriev's theorem)**.** *Suppose $d = 2$ and T is arbitrarily chosen. The following statements are equivalent:*

1) NA^w.

2) $\overline{\mathcal{A}_0^T} \cap L^1(\mathbb{R}_+^2, \mathcal{F}_1) = \{0\}$.

3) There exists a CPS.

In [15, Section 3], some counterexamples show that $\mathcal{A}_0^T$ is not necessarily closed under NA^w. In that case, it is not possible a priori to characterize the set of all super-hedging prices of a payoff.

Proposition 2.13. *Suppose that NA^w holds and $\mathcal{A}_0^T$ is closed. Consider a payoff $\xi_T \in L^1(\mathbb{R}^d, \mathcal{F}_T)$. Then, the set of all super-replicating prices $\Gamma(\xi_T)$ of ξ_T is given by*

$$\Gamma(\xi_T) = \left\{x_0 \in \mathbb{R}^d : \; x_0 Z_0 \geq E(Z_T \xi_T), \; \forall Z \,\text{CPS in}\, L^\infty\right\}.$$

Proof. Let us consider $x_0 \in \Gamma(\xi_T)$, i.e. there exists $V_T \in \mathcal{A}_0^T$ such that $x_0 + V_T = \xi_T$. We have $V_t = -\sum_{u=0}^t g_u$ where $g_u \in L^0(G_u, \mathcal{F}_u)$, $u = 0, \cdots, t$ and $t \leq T$. We have $Z_T \xi_T = Z_T(x_0 + V_T) = Z_T(x_0 + V_{T-1} - g_T)$. As $Z_T \in G_T^*$, we deduce that $Z_T \xi_T \leq Z_T(x_0 + V_{T-1})$ hence, considering the generalized conditional expectation, we get that

$$\begin{aligned} E(Z_T \xi_T | \mathcal{F}_{T-1}) &\leq Z_{T-1}(x_0 + V_{T-1}) \\ &= Z_{T-1}(x_0 + V_{T-2} - g_{T-1}) \\ &\leq Z_{T-1}(x_0 + V_{T-2}). \end{aligned}$$

Repeating the reasoning, i.e. take the successive generalized conditional expectations, we finally get that $E(Z_T \xi_T) \leq Z_0 x_0$.

Reciprocally, consider $x_0 \in \mathbb{R}^d$ such that $E(Z_T \xi_T) \leq Z_0 x_0$ for all CPS Z. Suppose by contradiction that

$$\xi_T - x_0 \notin \mathcal{A}_0^T \cap L^1(\mathbb{R}^d, \mathcal{F}_T).$$

By the Hahn-Banach separation theorem, there exists $\hat{Z} \in L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ and $c \in \mathbb{R}$ such that

$$E(\hat{Z} X) < c < E(\hat{Z}(\xi_T - x_0)), \quad \forall \text{CPS}\, Z.$$

As $\mathcal{A}_0^T$ is a cone, we deduce that $E(\hat{Z} X) \leq 0$ for all $X \in \mathcal{A}_0^T$ and $c > 0$. With $X = -g_t \in L^0(-G_t, \mathcal{F}_t) \subseteq \mathcal{A}_0^T$, we have $E(\hat{Z}_t g_t) \geq 0$ for any $g_t \in L^0(G_t, \mathcal{F}_t)$, where $\hat{Z}_t = E(\hat{Z} | \mathcal{F}_t)$. Arguing by contradiction with a measurable selection argument, see [15, Theorem 5.4.1, Section 5.4], we deduce that $\hat{Z}_t \in G_t^*$ a.s. Let us define $Z_t = \alpha \bar{Z}_t + \hat{Z}_t$ where Z is a CPS. By construction Z is a CPS if $\alpha > 0$. Moreover, with α small enough, we get that $E(\hat{Z}(\xi_T - x_0)) > 0$ in contradiction with the property satisfied by x_0. □

In the literature, a stronger no-arbitrage condition NA^r has been introduced. This condition means that there is no arbitrage opportunity even if the transaction costs are slightly smaller. Equivalently, this means that there is no arbitrage opportunity when the solvency set is larger, i.e. the positive dual is smaller. A CPS for this enlarged market is therefore in the interior of the initial positive dual. This ensures the closedness of $\mathcal{A}_0^T$, see [15, Section 3.2.2], so that Proposition 2.13 applies. This condition NA^r appears to be crucial to derive a FTAP as in the papers [2], [20] and [8] among others.

2.2 A new approach based on the liquidation value

In the last section, we have seen that the set of all terminal claims $\mathcal{A}_0^T$ is not necessarily closed under NA^w. A natural question is to understand whether this is the case for the liquidation values of these terminal claims. We consider here the case $d = 2$. The first asset is riskless and defined by the price $S_t^0 = 1$, $t = 0, \cdots, T$. The risky asset is defined by the bid and ask prices S_t^b and S_t^a such that $0 < S_t^b \leq S_t^a$, $t = 0, \cdots, T$. At time t, S_t^b and S_t^a are respectively the prices we get when selling/buying one unit of the risky asset. That corresponds to the best bid/ask prices in an order book. The liquidation value process is then:

$$L_t((x, y)) = x + y^+ S_t^b - y^- S_t^a, \quad t = 0, \cdots, T.$$

We then define $G_t := \{z \in \mathbb{R}^2 : \; L_t(z) \geq 0\}$ as in the last section. Similarly, we define

$$\begin{aligned} \mathcal{A}_u^t : &= \sum_{r=u}^t L^0(-G_r, \mathcal{F}_r), \\ \mathcal{L}_u^t : &= \{L_t(V_t) : \; V_t \in \mathcal{A}_u^t\} \quad 0 \leq u \leq t \leq T. \end{aligned}$$

In the following, we consider a technical condition:

E: For $T \geq 2$, for all $t \leq T-1$ and $u \geq t+1$, $F_u \in \mathcal{F}_u$,

(i) If $S_t^a = S_t^b$ on F_u, then there exists $r \geq u$ such that $S_t^a \geq S_r^a$ on F_u.

(ii) If $S_t^b = S_t^a$ on F_u, then there exists $r \geq u$ such that $S_r^b \geq S_t^b$ on F_u.

In [27], we provide classical examples where Condition E is satisfied. Clearly, it is satisfied under the efficient market hypothesis, i.e. when $S^b < S^a$. The following is easy to prove:

Lemma 2.14. *The condition NA^w is equivalent to one of the equivalent conditions:*

$$\mathcal{L}_0^T \cap L^0(\mathbb{R}_+, \mathcal{F}_T) = \{0\} \Leftrightarrow \mathcal{A}_0^T \cap L^0(\mathbb{R}_+^2, \mathcal{F}_T) = \{0\}.$$

The following theorem is new and proved in [27]. It may be seen as an analog of the DMW Theorem 1.20.

Theorem 2.15. *Suppose that condition E holds for 3 steps or more. Then, the following conditions are equivalent:*

(i) NA^w.

(ii) $\mathcal{L}_0^T$ *is closed in probability and* $\mathcal{L}_0^T \cap L^0(\mathbb{R}_+, \mathcal{F}_T) = \{0\}$.

(iii) There exists $Q \sim P$ satisfying

$$dQ/dP \in L^\infty((0, \infty), \mathcal{F}_T)$$

such that $E_Q(L_T(V_T)) \leq 0$ for all $L_T(V_T) \in \mathcal{L}_0^T \cap L^1(\mathbb{R}, \mathcal{F}_T)$.

Proof. We provide here the proof in the case where there are only two steps. The general case is deduced by induction, see [27].

The implication $(ii) \Rightarrow (iii)$ follows from the Hahn-Banach separation theorem, see the last sections. The implications $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (i)$ are trivial.

Closedness. It remains to show that $(i) \Rightarrow (ii)$, i.e. $\mathcal{L}_0^T$ is closed in probability. With one time step, this is immediate as $\mathcal{L}_T^T = -L^0(\mathbb{R}_+, \mathcal{F}_T)$. We may show that, for any $\gamma \in \mathcal{L}_0^T$, $\gamma e_1 = -g_0^T \in \mathcal{A}_0^T$ where g_u^t, $u \le t$, is a general notation we introduce for the sequel to designate a sum $g_u^t = \sum_{r=u}^t g_r$ with $g_r \in L^0(\mathbf{G}_r, \mathcal{F}_r)$, $r \le T$. When considering a sequence of such elements, we write them $g_u^{t,n} = \sum_{r=u}^t g_r^n$ with $g_r^n \in L^0(\mathbf{G}_r, \mathcal{F}_r)$. In the following, we may suppose w.l.o.g. that $g_r \in \partial \mathbf{G}_t := \mathbf{G}_t \setminus \mathrm{int}\mathbf{G}_t$ for all $t \le T-1$. To do so, we withdraw $L_r(g_r) \ge 0$ from g_r that we add to G_T. Recall that, by the Grigoriev theorem, there exists a CPS Z.

Two steps. Consider $\gamma_T^\infty = \lim_n \gamma_T^n$ where $\gamma_T^n e_1 = -g_{T-1}^{T,n} = -g_{T-1}^n - g_T^n \in \mathcal{L}_{T-1}^T$. Define the set

$$\Gamma_{T-1} := \{\liminf |g_{T-1}^n| = \infty\} \in \mathcal{F}_{T-1}.$$

Up to a random subsequence, we may suppose that $|g_{T-1}^n| > 0$. We normalize the sequences by setting $\tilde{\gamma}_T^n := \gamma_T^n / |g_{T-1}^n|$, $\tilde{g}_{T-1}^n := g_{T-1}^n / |g_{T-1}^{T,n}|$, $\tilde{g}_T^n := g_T^n / |g_{T-1}^{T,n}|$. As $|\tilde{g}_{T-1}^n| = 1$, we may assume that $\tilde{g}_{T-1}^n \to \tilde{g}_{T-1}^\infty \in \mathbf{G}_{T-1}$, see [15, Lem. 2.1.2]. As $\lim_n \tilde{\gamma}_T^n e_1 = 0$, we deduce that $\tilde{g}_T^n \to \tilde{g}_T^\infty \in \mathbf{G}_T$ and $\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty = 0$ where $\tilde{g}_{T-1}^\infty \in \partial \mathbf{G}_{T-1}$ and $\tilde{g}_T^\infty \in \mathbf{G}_T$. We set $\tilde{g}_{T-1}^\infty = \tilde{g}_T^\infty = 0$ on $\Lambda_{T-1} = \Omega \setminus \Gamma_{T-1} \in \mathcal{F}_{T-1}$. Let Z be a CPS. As $Z_T(\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty) = 0$, we deduce that $Z_{T-1}\tilde{g}_{T-1}^\infty + \mathbb{E}(Z_T \tilde{g}_T^\infty | \mathcal{F}_{T-1}) = 0$. By duality, i.e. using the property that a CPS evolves in the positive dual of $\mathbf{G}$, we get that $Z_{T-1}\tilde{g}_{T-1}^\infty = Z_T \tilde{g}_T^\infty = 0$. As $\tilde{g}_T^\infty = -\tilde{g}_{T-1}^\infty$ is $\mathcal{F}_{T-1}$-measurable, we get that

$$0 = \mathbb{E}(Z_T \tilde{g}_T^\infty | \mathcal{F}_{T-1}) = Z_{T-1} \tilde{g}_T^\infty.$$

So, $Z_{T-1}\tilde{g}_T^\infty = Z_T \tilde{g}_T^\infty$ hence $Z_{T-1} \in (\mathbb{R}_+ Z_T) \cap \mathbf{G}_T^*$. Therefore,

$$Z_{T-1}\gamma_T^n e_1 = -Z_{T-1} g_{T-1}^n - Z_{T-1} g_T^n \le 0$$

by duality. Since $Z_{T-1} e_1 > 0$, we deduce that $\gamma_T^n \le 0$. So, $\gamma_T^n e_1 = -\hat{g}_{T-1}^{T,n}$ a.s., where $\hat{g}_{T-1}^n = g_{T-1}^n 1_{\Lambda_{T-1}} \in \partial \mathbf{G}_{T-1}$ and $\hat{g}_T^n = g_T^n 1_{\Lambda_{T-1}} + (-\gamma_T^n e_1) 1_{\Gamma_{T-1}}$ belongs to $L^0(\mathbf{G}_T, \mathcal{F}_T)$. By construction, $\liminf_n |\hat{g}_{T-1}^n| < \infty$ hence we may suppose that $\hat{g}_{T-1}^n \to \hat{g}_{T-1}^\infty \in L^0(\mathbf{G}_{T-1}, \mathcal{F}_{T-1})$ by [15, Lem. 2.1.2]. We deduce that $\hat{g}_T^n \to \hat{g}_T^\infty \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ hence $\gamma_T^\infty = -\hat{g}_{T-1}^{T,\infty} \in \mathcal{L}_{T-1}^T$. □

In the following, we denote by $\mathcal{M}^\infty(P)$ the set of all $Q \sim \mathrm{P}$ such that $dQ/d\mathrm{P} \in L^\infty$ and $\mathbb{E}_Q L_T(V) \le 0$ for all $L_T(V) \in \mathcal{L}_0^T$. For any contingent claim $\xi \in L^1(\mathbb{R}, \mathcal{F}_T)$, we define the set $\Gamma(\xi)$ of all initial endowments of portfolio processes whose terminal liquidation values coincide with ξ, i.e.

$$\Gamma(\xi) := \{x \in \mathbb{R} : \ \exists V \in \mathcal{A}_0^T : \ L_T(xe_1 + V_T) = \xi\}.$$

Corollary 2.16. *Suppose that condition* **E** *holds. Let* $\xi \in L^0(\mathbb{R}, \mathcal{F}_T)$ *be such that* $\mathbb{E}_{\mathrm{P}}|\xi| < \infty$. *Then, under condition* NA^w, $\Gamma_\xi = [\sup_{Q \in \mathcal{M}^\infty(P)} \mathbb{E}_Q \xi, \infty)$.

The proof is very similar to the one for frictionless markets.

2.3 When the solvency set is not a convex cone

All the arguments we have used in the previous sections are possible because the solvency sets are closed convex sets. This allows to deduce a dual characterization of no-arbitrage conditions and super-hedging prices. In particular, $\mathcal{A}_0^T$ is a closed convex cone. Clearly, this classical principle in mathematical finance is no more valid if G is not convex. In the following, we present a modest new contribution allowing to compute the super-hedging prices in a non convex setting.

Let us consider the very simple example with two assets and two time steps. The first one is $S_t^0 = 1$, $t = 0, 1$, the second one is risky and defined by the price S_t, $t = 0, 1$. We suppose that there are proportional transaction costs to pay when buying/selling the risky asset. Moreover, a fixed cost $c \ge 0$ is charged. We suppose that the agent accepts to pay c only if the liquidation value of the risky position y is either negative or larger than the fixed cost. Indeed, if $0 < yS_1(1-\epsilon) \le c$, it is not interesting for the agent to liquidate the risky position y. Precisely, we suppose that

$$\begin{aligned} & L_t((x,y)) \\ = \ & x + (yS_t(1-\epsilon) - c)^+ - y^- S_t(1+\epsilon) - c1_{y<0}, \end{aligned}$$

for $t = 0, 1$. As usual, we define the solvency set $G_t := \{z \in \mathbb{R}^2 : \ L_t(z) \ge 0\}$, $t = 0, 1$. We may easily observe that G is not convex. By [28], $z \mapsto L_t(z)$ is upper semi-continuous.

A new approach is necessary to obtain the super-hedging prices of some payoff $\xi_1 \in L^0(\mathbb{R}e_1, \mathcal{F}_1)$. We suppose that ξ_1 is of the form $\xi_1 = h(S_1)e_1$ where h is a continuous function. The problem we propose to solve is to characterize the set of all prices $p_0 \in \Gamma(h)$ such that $p_0 e_1 - g_0 - g_1 = h(S_1)e_1$ for some $g_t \in L^0(G_t, \mathcal{F}_t)$, $t = 0, 1$, i.e.

$$\Gamma(h) = \{p_0 \in \mathbb{R} : \quad p_0 - h(S_1) + L_1(-g_0) \ge 0\}.$$

Notice that $p_0 \in \Gamma(h)$ if and only if $p_0 \ge p_0(g_0) = \operatorname{ess\,sup}_{\mathcal{F}_0}(h(S_1) - L_1(-g_0))$, where the notion of essential supremum is given in [15, Section 5.3.1]. Moreover,

$\Gamma(h)$ is an interval. Our goal is to determine $\inf \Gamma(h)$. By [1, Proposition 2.13], we have

$$p_0(g_0) = \sup_{s \in \mathrm{supp}\,(S_1)} g(g_0, s), \quad g_0 \in G_0,$$

where supp (S_1) is the support of S_1 and

$$\begin{aligned} g(g_0, s) &= h(s) + \gamma(g_0, s), \quad g_0 = (x_0, y_0), \\ \gamma((x_0, y_0), s) &= x_0 - (y_0 s(1-\epsilon) + c)^- \\ &\quad + y_0^+ s(1+\epsilon) + c1_{y_0>0}. \end{aligned}$$

In the following, we suppose that $h(s) = (s-K)^+$, $K \geq 0$, and we use the notation $g_0 = (x_0, y_0)$. We suppose that supp $(S_1) = [S_1^{\min}, S_1^{\max}]$. We have:

$$g(g_0, s) = \begin{cases} g^1(g_0, s) = x_0 + y_0 s(1+\epsilon) + c + (s-K)^+, \\ \qquad \text{for } y_0 > 0, \\ g^2(g_0, s) = x_0 + (s-K)^+, \\ \qquad \text{for } 0 < s \leq \frac{-c}{y_0(1-\epsilon)},\ y_0 < 0 \\ g^3(g_0, s) = x_0 + y_0 s(1-\epsilon) + c + (s-K)^+, \\ \qquad \text{for } s > \frac{-c}{y_0(1-\epsilon)},\ y_0 < 0. \end{cases}$$

Therefore,

$$p_0(g_0) = \begin{cases} g^1(g_0, S_1^{\max}), & \text{for } y_0 > 0, \\ g^2\left(g_0, \frac{-c}{y_0(1-\epsilon)} \vee S_1^{\min}\right), & \text{for } y_0 \leq \frac{-1}{1-\epsilon} \\ \max\left(g^2\left(g_0, \frac{-c}{y_0(1-\epsilon)} \vee S_1^{\min}\right), g^3(g_0, S_1^{\max})\right), & \\ \qquad \text{for } 0 > y_0 > \frac{-1}{1-\epsilon} \end{cases}$$

Notice that $g_0 = (x_0, y_0) \in G_0$ if and only if $x_0 + L_0((0, y_0)) \geq 0$, i.e. $x_0 \geq \delta(y_0) := -L_0((0, y_0))$. Therefore,

$$p_0^* = \inf \Gamma(h) = \inf_{y_0 \in \mathbb{R}} p_0(\delta(y_0), y_0).$$

When computing p_0^*, we obtain the argmin y_0 and $x_0 = \delta(y_0)$ such that $g_0 = (x_0, y_0)$. For instance, with $c = 1.5$, $\epsilon = 5\%$ and $K = 50$, we get that $g_0 = (64.05, -0.61)$ and $p_0^* = 74$. With $c = \epsilon = 0$ and $K = 50$, we get $g_0 = (56.99, -0.5699)$ and $p_0^* = 65.27$. In Table of Figure 2, minimal prices are computed.

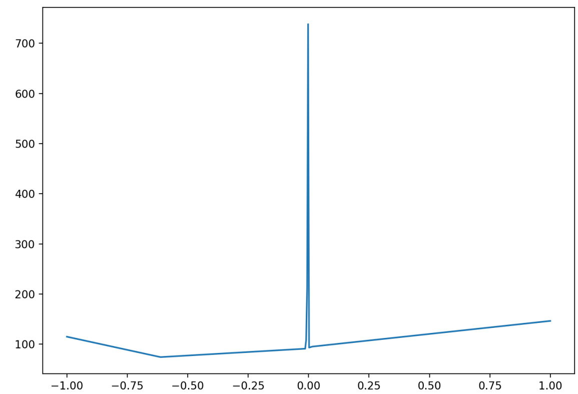

Fig. 1: The price function $y_0 \mapsto p_0(\delta(y_0), y_0)$ for $y_0 \in [-1, 1]$. The parameters are $c = 1.5$, $K = 50$, $\epsilon = 5\%$.

	K=30	K=50	K=70	K=100
$c = \epsilon = 0\%$	$p_0^* = 85.27$	$p_0^* = 65.27$	$p_0^* = 49.96$	$p_0^* = 28.21$
$c = 1.5,\ \epsilon = 0\%$	$p_0^* = 85.72$	$p_0^* = 66.8$	$p_0^* = 50.28$	$p_0^* = 28.8$
$c = 1.5,\ \epsilon = 1\%$	$p_0^* = 88$	$p_0^* = 68$	$p_0^* = 49.22$	$p_0^* = 34.5$
$c = 1.5,\ \epsilon = 5\%$	$p_0^* = 91.8$	$p_0^* = 74$	$p_0^* = 56.8$	$p_0^* = 31.15$
$c = 1.5,\ \epsilon = 10\%$	$p_0^* = 98.8$	$p_0^* = 79.2$	$p_0^* = 60$	$p_0^* = 34.1$

Fig. 2: Numerical computation of the minimal prices for several parameters.

3 Conclusion

We have discovered the main arguments and tools allowing to characterize no-arbitrage conditions and then deduce dual characterizations of super-hedging prices. It was possible to do it because the set of terminal claims is a closed convex cone under NA or other stronger condition. In practice, the transaction costs are not necessarily linear so that the solvency set G is not a cone. Then, new approaches need to be invented. One of them could be to use the natural stochastic preorder generated by G, i.e. $x \geq_{G_t} y$ if and only if $x - y \in G_t$, see [28] and [29]. We also presented a new approach that should be generalized.

For readers who wish to deepen their knowledge on arbitrage theory, a list of references is given in the bibliography. Among the very well known authors currently working on arbitrage theory[2], we may cite Bouchard B, Campi L., Cherny A., Delbaen F. , Guasoni P., Kabanov Y. , Rásonyi M., Schachermayer W. and Touzi N.

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Emmanuel Lepinette

Paris Dauphine university, PSL research university, Ceremade, CNRS, UMR, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France, and
Gosaef, Faculty of Sciences of Tunis, Tunisia.

E-mail: emmanuel.lepinette@ceremade.dauphine.fr