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Area and perimeter foliations on spaces of polygons

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Abstract

We describe all families of star-shaped n-polygons in the Euclidean plane with prescribed perimeter and area; they are leaves of a foliation $\mathcal{F}$ on the space $\mathcal{P}_n^*$ of star-shaped polygons. In so doing, we study and record some geometric properties of convex polygons, for instance their inscriptibility in a circle and their regularity in relation with perimeter and area.

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1 Introduction

Let $\mathbb{E}$ be an Euclidean vector plane equipped with its underlying affine structure and an orientation given by an orthonormal basis. The word 'isometry' means a transformation of $\mathbb{E}$ which preserves the distance; necessarily it is an affine transformation of $\mathbb{E}$. The origin of $\mathbb{E}$ is denoted O. For $A, B \in \mathbb{E}$ we denote $\overrightarrow{AB}$ the vector $B - A$.

A *figure* in the plane usually refers to part of the plane that has a certain peculiarity: we see it all (it is bounded) or we have a global understanding of how it is shaped even when it escapes our view, like a straight line. A polygon is such a figure: it is bounded and bordered by a finite number of segments called sides or edges. Each polygon has a perimeter and an area associated to it, and these basic numerical invariants shed interesting light on the geometry of families of polygons, which is the main topic of this work.

The polygons of the plane are numerous and their shapes and sizes are varied. So the question of their equivalence therefore arises naturally. But in which sense?

From the set point of view, two polygons $\wp$ and $\wp'$ are always equivalent: there is a bijection of $\mathbb{E}$ which sends one on the other. However, we would like this bijection to preserve properties related to the affine Euclidean structure. Depending on the number of properties we wish to preserve, there are several notions of equivalence. We list those that are of direct interest to us.

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We will say that $\wp$ and $\wp'$ are:

1. *isometric* if there exists an isometry $f : \mathbb{E} \longrightarrow \mathbb{E}$ such that $f(\wp) = \wp'$. We can superimpose them; and we can still do that without going out of the plane if $\wp$ and $\wp'$ are *directly isometric*, that is f preserves the orientation;
2. *similar* if there is a similarity $f : \mathbb{E} \longrightarrow \mathbb{E}$ such that $f(\wp) = \wp'$. In a way, one of them is an enlargement of the other (as for the photos);
3. *equivalent* (more on this shortly) if they have the same area. In this case, we can go from one to another by geometric cutting and gluing;
4. *isoperimetric* if they have the same perimeter.

The isometric equivalence is the strongest and implies all the others. So it is too rigid to be 'useful': two isometric polygons differ only in the positions they occupy in the plane. It is rather the equivalences (3) and (4) that will occupy us here.

There is nowadays a vast literature on the topology and geometry of spaces of polygons up to isometry or up to similarity, with an increased interest in the topic in more recent years (see in particular [2] and [4]). The spaces investigated in the literature turn out to be in general compact manifolds and their cohomology has been studied. In this paper we take a slightly distinct point of view where we use classical invariants to foliate spaces of polygons (more precisely those that are star-shaped).

This project was motivated by a question of Geoffrey Letellier: *Are there two non-isometric triangles with same perimeter and same area*[1]? It led us first to the construction of a foliation on the space of triangles: each leaf consists of triangles having same perimeter and same area. Then we made a more general study

[1] Letellier answered his own question by constructing a one parameter family of isosceles triangles having the same perimeter and the same area (his example is given in subsection 5.4). Valerio Vassallo relayed Letellier's question to the first author who noticed the relevant foliation $\mathcal{F}$ on the space of triangles.

for the space of star-shaped polygons; this space contains convex polygons for which some properties related to area and perimeter have also been studied. Some of them are certainly known, but not easily accessible in the 'geometric literature', which motivated us to include them in the final section 6.

All integers n we consider in this paper will be greater than or equal to 3.

2 Preliminaries

Definition 2.1. *A non degenerate polygon of* $\mathbb{E}$ *is an element* $\wp = (M_1, \cdots, M_n)$ *of* $\mathbb{E}^n$ *such that:*

1. *for* $i \neq j$ *the point* M_i *is distinct from* M_j*;*
2. *for any* $k \in \{1, \cdots, n\}$*, the oriented angle* $\widehat{M_k} = (\overrightarrow{M_k M_{k+1}}, \overrightarrow{M_k M_{k-1}})$ *has its measure in* $]0, 2\pi[\setminus\{\pi\}$.

By convention, $M_0 = M_n$ and $M_{n+1} = M_1$. The orientation of the angle $\widehat{M_k}$ is the same as the orientation of the trigonometric circle centered at the point M_k.

The points M_k and the segments $[M_k M_{k+1}]$ are respectively the *vertices* and the *sides* of the polygon $\wp$. If M_i and M_j are two non successive vertices, that is $|i-j| > 1$, we say that the segment $M_i M_j$ is a *diagonal* of the polygon.

Recall that a polygon is said to be:

- *equilateral* if all its sides have the same length;
- *inscribable* (or cyclic) if all its vertices are on a same circle;
- *regular* if it is both inscribable and equilateral.

The set of all n-polygons of the plane $\mathbb{E}$ will be denoted by $\widetilde{\mathcal{P}}_n$. Let f be an isometry of the Euclidean plane $\mathbb{E}$. The image by f

$$\wp' = (M'_1, \cdots, M'_n) = (f(M_1), \cdots, f(M_n))$$

of a polygon $\wp = (M_1, \cdots, M_n)$ is a polygon of $\mathbb{E}$ such that:

$$\widehat{M'_k} = \widehat{M_k} \quad \text{and} \quad ||\overrightarrow{M'_k M'_\ell}|| = ||\overrightarrow{M_k M_\ell}|| \quad \text{for} \quad (k, \ell) \in \{1, \cdots, n\}^2 \tag{2.1}$$

So we have a natural action:

$$\begin{array}{ccc} \mathrm{Isom}(\mathbb{E}) \times \widetilde{\mathcal{P}}_n & \longrightarrow & \widetilde{\mathcal{P}}_n \\ \Big(f, (M_1, \cdots, M_n)\Big) & \longmapsto & \Big(f(M_1), \cdots, f(M_n)\Big) \end{array} \tag{2.2}$$

where $\mathrm{Isom}(\mathbb{E})$ is the group of the affine isometries of $\mathbb{E}$. The quotient space of this action will be denoted:

$$\mathcal{P}_n = \widetilde{\mathcal{P}}_n / \mathrm{Isom}(\mathbb{E}) \tag{2.3}$$

The elements of $\mathcal{P}_n$ are called *geometric polygons* of $\mathbb{E}$.

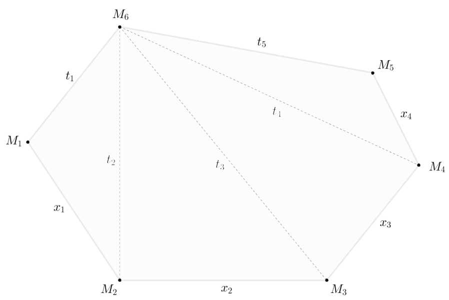
A geometric polygon of $\mathbb{E}$ is said to be *equilateral* (resp. *inscribable*) if it admits a representative which is equilateral (resp. inscribable).

Notation 2.1. *For an element* $\wp = (M_1, \cdots, M_n)$ *of* $\widetilde{\mathcal{P}}_n$*, we will use the following notations:*

- $\langle M_1, \cdots, M_n \rangle$ *is the equivalence class of* $(M_1, \cdots, M_n)$ *in* $\mathcal{P}_n$*,*
- *for* $k = 1, \cdots, n-2$*,* $x_k = M_k M_{k+1} = ||\overrightarrow{M_k M_{k+1}}||$.
- *for* $k = 1, \cdots, n-1$*,* $t_k = M_n M_k = ||\overrightarrow{M_n M_k}||$.

The positive numbers $t_1, x_1, \cdots, x_{n-2}, t_{n-1}$ are the lengths of the sides and $t_2, t_3, \cdots, t_{n-2}$ are the lengths of the diagonals from the vertex M_n (see the picture bellow for the case of the hexagon). We have $(n-2)$ lengths of type x_k and $(n-1)$ lengths of type t_k.

Since an isometry preserves the distance in the Euclidean plane $\mathbb{E}$, $(t_1, x_1, t_2, x_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1})$ does not depend on the choice of representative $(M_1, \cdots, M_n)$ of the geometric polygon $\langle M_1, \cdots, M_n \rangle$.



For $1 \leq k \leq n-2$, the lengths t_k, x_k, t_{k+1} of the triangle (M_k, M_n, M_{k+1}) satisfy the inequalities:

$$\begin{array}{l} 0 < t_k < x_k + t_{k+1} \\ 0 < x_k < t_k + t_{k+1} \\ 0 < t_{k+1} < x_k + t_k \end{array}$$

that is; (t_k, x_k, t_{k+1}) is an element of the open set $\mathcal{V}$ of $\mathbb{R}^3$ consisting of the triplets (x, y, z) satisfying:

$$\begin{array}{l} 0 < x < y + z \\ 0 < y < x + z \\ 0 < z < x + y. \end{array}$$

Moreover, one can verify by induction that, for the lengths of the sides of a polygon, each length is strictly smaller than the sum of the others. For instance, we have the following nice [6]:

Theorem 2.1. *For any natural integer* $n \geq 3$ *and any* n*-tuple* $u = (u_1, \cdots, u_n)$ *of positive real numbers, there exists a unique inscribable polygon* $\langle M_1, \cdots, M_n \rangle$ *such that* $M_k M_{k+1} = u_k$ *for* $k \in \{1, \cdots, n\}$ *if and only if, for any* $j \in \{1, \cdots, n\}$*, we have the inequality:*

$$u_j < \sum_{k \neq j} u_k.$$

Recall that a polygon $\wp = (M_1, \cdots, M_n)$ of $\mathbb{E}$ is *convex* if, for any $k \in \{1, \cdots, n\}$, the oriented angle $\widehat{M_k} = (\overrightarrow{M_kM_{k+1}}, \overrightarrow{M_kM_{k-1}})$ has its measure in $]0, \pi[$. But for our purposes, we only need a slightly weaker notion.

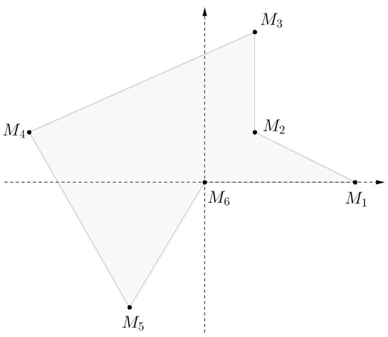
Definition 2.2. A polygon $\wp = (M_1, \cdots, M_n)$ of $\mathbb{E}$ is said to be *star-shaped polygon* with respect to the vertex M if, for any vertex $N \in \{M_1, \cdots, M_n\}$ not adjacent to M, the open segment $]MN[$ is contained in the interior of $\wp$. Of course, any convex polygon is star-shaped with respect to any of its vertices.

Star-shaped polygons $(M_1, \cdots, M_n)$ with respect to the vertex M_n form a subspace $\widetilde{\mathcal{P}}_n^*$ of $\widetilde{\mathcal{P}}_n$, invariant under the action of Isom($\mathbb{E}$) (on $\widetilde{\mathcal{P}}_n$).

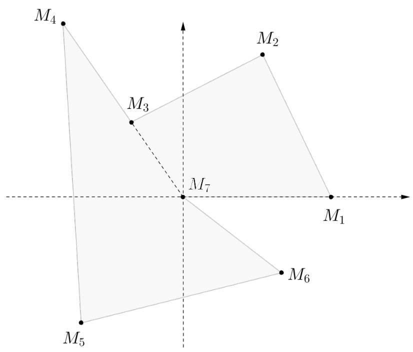
One can easily see that any element of $\widetilde{\mathcal{P}}_n^*$ is isometric to a unique star-shaped polygon $\wp = (M_1, \cdots, M_n)$ with respect to M_n such that:

- $M_n = M_0 = O$ the origin of $\mathbb{E}$,
- M_1 has coordinates $(t_1, 0)$ with $t_1 > 0$
- for any $k \in \{1, \cdots, n-2\}$, the measure of the angle $\widehat{M_kOM_{k+1}} = (\overrightarrow{OM_k}, \overrightarrow{OM_{k+1}})$ is in $]0, \pi[$.

$\mathcal{P}_n^*$ will denote the space of the star-shaped polygons satisfying these conditions.



Allowed



The measure of the angle $\widehat{M_3M_7M_4}$ is 0.
So it is not allowed.

Next, we will recall some metric properties in a triangle which will be very useful for determining the space of polygons on which we will define area and perimeter foliations in section 4.

Let OMN be a non degenerate triangle ; we set $OM = t$, $ON = s$, $MN = x$ and $\alpha = \widehat{MON}$.

All the real numbers t, s, x are positive and $\alpha \in]0, \pi[$. For the triangle OMN we denote $r > 0$ the radius of its inscribed circle, p its perimeter and a its area. We have the following well known formulas:

$$p = t + x + s$$

$$a = \frac{1}{4}\sqrt{(t+x+s)(t-x+s)(t+x-s)(-t+x+s)}$$

(Héron's formula), $r = \frac{a}{p}$ and $\cos(\alpha) = \frac{t^2+s^2-x^2}{2st}$ (cosine's law of Al-Kashi).

We consider The continuous function

$$\alpha : \mathcal{V} \longrightarrow]0, \pi[, \ (t, x, s) \mapsto \arccos \frac{t^2 + s^2 - x^2}{2st}$$

and the open set Ω_n of $(\mathbb{R}_+^*)^{2n-3}$ formed of points $(t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1})$ such that $(t_k, x_k, t_{k+1}) \in \mathcal{V}$, for $k = 1, \cdots, n-2$ and $\sum_{k=1}^{n-2} \alpha(t_k, x_k, t_{k+1}) < 2\pi$.

Let $\mathcal{L}_n : \mathcal{P}_n^* \longrightarrow \Omega_n$ be the map

$$\begin{aligned} \mathcal{L}_n(\wp) &= \mathcal{L}_n(\langle M_1, \cdots, M_n \rangle) \\ &= (t_1, x_1, t_2, x_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \end{aligned}$$

where the real numbers $t_1, x_1, t_2, x_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}$ are given by the formulas in Notation 2.1.

From now on we will identify the geometric star-shaped polygons to the points of the open set Ω_n by the map $\mathcal{L}_n$. This identification $\mathcal{P}_n^* \simeq \Omega_n$ will enable one to study easily some properties of the space of geometric star-shaped polygons.

Let $\varphi : X \to Y$ be any map. A nonempty subset of X of the form $\varphi^{-1}(\{y\})$ will be called the *level set* (level line, level surface, level manifold...) of φ at level $y \in Y$.

3 The geometric inscribable polygons for $n \geq 4$

First note that, following our definition 1.3, an inscribable polygon $(M_1, \cdots, M_n)$ is always convex. It is therefore a star-shaped polygon with respect to any of its vertices. We can therefore represent it by an element of $\mathcal{P}_n^*$. In this way, the set Γ_n of inscribable polygons of $\mathbb{E}$ can be viewed as a subset of the open set Ω_n. (Recall that any regular geometric polygon is inscribable.)

Remark 1. The fact that a polygon $(M_1, \cdots, M_n)$ is inscribable is equivalent to the fact that each one of the $n-3$ quadrilaterals: $Q_1 = (M_n, M_1, M_2, M_3), \cdots$, $Q_{n-3} = (M_n, M_{n-3}, M_{n-2}, M_{n-1})$ is inscribable.

Lemma 3.1. *A convex quadrilateral* (A, B, C, D) *is inscribable if and only if the distances* $a = AB$, $b = BC$, $c = CD$, $d = DA$ *and* $e = BD$ *satisfy the relation:*

$$ad(b^2 + c^2 - e^2) + bc(a^2 + d^2 - e^2) = 0. \quad (3.1)$$

Proof. Let α and β denote the measures respectively of the angles $\widehat{A}$ and $\widehat{C}$. Then (by the cosine's law of Al-Kashi): $0 < \alpha < \pi$, $0 < \beta < \pi$ and

$$a^2 + d^2 - 2ad\cos\alpha = e^2 = b^2 + c^2 - 2bc\cos\beta.$$

We deduce:

$$\cos\alpha = \frac{a^2 + d^2 - e^2}{2ad} \quad \text{and} \quad \cos\beta = \frac{b^2 + c^2 - e^2}{2bc}$$

On the other hand, the quadrilateral (A, B, C, D) is inscribable if and only if $\alpha + \beta = \pi$ or, equivalently, if $\cos\beta = -\cos\alpha$. Hence: (A, B, C, D) is inscribable if and only if

$$\frac{b^2 + c^2 - e^2}{2bc} = -\frac{a^2 + d^2 - e^2}{2ad}$$

which is also equivalent to the relationship:

$$ad(b^2 + c^2 - e^2) + bc(a^2 + d^2 - e^2) = 0$$

□

Remark 1 and Lemma 3.1 make it possible to realize Γ_n as a level set of a differentiable map. More precisely, consider the maps $\Theta : \mathbb{R}^5 \longrightarrow \mathbb{R}$, $\gamma_k : \Omega_n \longrightarrow \mathbb{R}$ and $\gamma : \Omega_n \longrightarrow \mathbb{R}^{n-3}$ defined by:

- $\Theta(u) = u_1u_2(u_4^2 + u_5^2 - u_3^2) + u_4u_5(u_1^2 + u_2^2 - u_3^2)$ for $u = (u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5$,
- $\gamma_k(\omega) = \Theta(t_k, x_k, t_{k+1}, x_{k+1}, t_{k+2})$ for $k \in \{1, \cdots, n-3\}$ and $\omega = (t_1, x_1, t_2, \cdots, x_{n-2}, t_{n-1}) \in \Omega_n$,
- $\gamma(\omega) = (\gamma_1(\omega), \cdots, \gamma_{n-3}(\omega))$.

Proposition 3.1. *The set* Γ_n *of geometric inscribable polygons is given by* $\Gamma_n = \gamma^{-1}(\{0\})$. *Moreover, this set is a differentiable submanifold of dimension* n *of the Euclidean space* $\mathbb{R}^{2n-3}$.

Proof. Let $\omega = (t_1, x_1, ..., t_{n-2}, x_{n-2}, t_{n-1})$ be an element of Ω_n represented by a polygon $(M_1, \cdots, M_n)$ (in $\mathcal{P}_n^*$). Then we have the following equivalences:

$$\begin{aligned}
& \omega \in \Gamma_n \\
\Longleftrightarrow\ & (M_n, M_k, M_{k+1}, M_{k+2}) \text{ is inscribable,} \\
& \text{for} 1 \leq k \leq n-3 \\
\Longleftrightarrow\ & \Theta(t_k, x_k, t_{k+1}, x_{k+1}, t_{k+2}) = 0, \\
& \text{for} 1 \leq k \leq n-3 \\
\Longleftrightarrow\ & \gamma_k(\omega) = 0, \text{ for } \leq k \leq n-3 \\
\Longleftrightarrow\ & \gamma(\omega) = 0 \\
\Longleftrightarrow\ & \omega \in \gamma^{-1}(\{0\}).
\end{aligned}$$

Now we will prove, by induction on $n \geq 4$, that the map γ has maximal rank at each point ω of Ω_n.

- **The case** $n = 4$. We have $\gamma = \gamma_1$ and the map $\gamma : \Omega_4 \longrightarrow \mathbb{R}$ defined on $\omega = (t_1, x_1, t_2, x_2, t_3)$ by:

$$\gamma(\omega) = t_1x_1(t_3^2 + x_2^2 - t_2^2) + t_3x_2(t_1^2 + x_1^2 - t_2^2)$$

has maximal rank at each point $\omega \in \Omega_4$. Indeed,

$$\frac{\partial\gamma}{\partial t_2}(\omega) = -2t_2(t_1x_1 + t_3x_2) \neq 0.$$

So $\Gamma_4 = \gamma^{-1}(\{0\})$ is a codimension 1 submanifold of the open set Ω_4 and then a submanifold of dimension 4 of $\mathbb{R}^5$.

- Heredity. Suppose that, for a fixed integer $n \geq 4$, the map γ has maximal rank at each point of Ω_n. If to each element $\omega = (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}, x_{n-1}, t_n) \in \Omega_{n+1}$ we associate:

$$\begin{aligned}
\omega' &= (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n \\
\omega'' &= (t_{n-2}, x_{n-2}, t_{n-1}, x_{n-1}, t_n) \in \Omega_4
\end{aligned}$$

then we can write $\omega = (\omega', x_{n-1}, t_n)$ and $\gamma(\omega) = (\gamma(\omega'), \gamma(\omega''))$. Note that here we are using the same notation γ for three different maps. The Jacobian matrix of the map $\gamma : \Omega_{n+1} \subset \mathbb{R}^{2n-1} \to \mathbb{R}^{n-2}$ at each point $\omega \in \Omega_{n+1}$ is given by:

$$\begin{pmatrix} A(\omega') & 0 \\ B(\omega'') & C(\omega'') \end{pmatrix} \quad (3.2)$$

where $A(\omega')$ is the matrix

$$\begin{pmatrix}
\frac{\partial\gamma_1}{\partial t_1}(\omega') & \frac{\partial\gamma_1}{\partial x_1}(\omega') & \cdots & \frac{\partial\gamma_1}{\partial t_{n-2}}(\omega') & \frac{\partial\gamma_1}{\partial x_{n-2}}(\omega') & \frac{\partial\gamma_1}{\partial t_{n-1}}(\omega') \\
\frac{\partial\gamma_2}{\partial t_1}(\omega') & \frac{\partial\gamma_2}{\partial x_1}(\omega') & \cdots & \frac{\partial\gamma_2}{\partial t_{n-2}}(\omega') & \frac{\partial\gamma_2}{\partial x_{n-2}}(\omega') & \frac{\partial\gamma_2}{\partial t_{n-1}}(\omega') \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial\gamma_{n-3}}{\partial t_1}(\omega') & \frac{\partial\gamma_{n-3}}{\partial x_1}(\omega') & \cdots & \frac{\partial\gamma_{n-3}}{\partial t_{n-2}}(\omega') & \frac{\partial\gamma_{n-3}}{\partial x_{n-2}}(\omega') & \frac{\partial\gamma_{n-3}}{\partial t_{n-1}}(\omega')
\end{pmatrix}$$

which is the Jacobian matrix of the differentiable map $\gamma : \Omega_n \subset \mathbb{R}^{2n-3} \to \mathbb{R}^{n-3}$ at the point ω',

$$B(\omega'') = \left(0, \cdots, 0, \frac{\partial\Theta}{\partial u_1}(\omega''), \frac{\partial\Theta}{\partial u_2}(\omega''), \frac{\partial\Theta}{\partial u_3}(\omega'')\right)$$

$$C(\omega'') = \left(\frac{\partial\Theta}{\partial u_4}(\omega''), \frac{\partial\Theta}{\partial u_5}(\omega'')\right).$$

By the induction hypothesis, the matrix $A(\omega')$ has rank $n-3$. On the other hand, the two partial derivatives

$$\begin{aligned}
\frac{\partial\Theta}{\partial u_4}(\omega'') &= 2x_{n-1}t_{n-2}x_{n-2} + t_n\left(x_{n-2}^2 + t_{n-2}^2 - t_{n-1}^2\right) \\
\frac{\partial\Theta}{\partial u_5}(\omega'') &= 2t_nt_{n-2}x_{n-2} + x_{n-1}\left(x_{n-2}^2 + t_{n-2}^2 - t_{n-1}^2\right)
\end{aligned}$$

do not vanish simultaneously since otherwise we will have:

$$0 = x_{n-1}\frac{\partial\Theta}{\partial u_4}(\omega'') - t_n\frac{\partial\Theta}{\partial u_5}(\omega'') = 2x_{n-1}t_{n-2}(x_{n-1}^2 - t_n^2)$$

which implies $x_{n-1} = t_n$. Taking into account this equality in the relation $\frac{\partial\Theta}{\partial u_4}(\omega'') = 0$ we obtain :

$$t_n\left[(x_{n-2}+t_{n-2})^2 - t_{n-1}^2\right] = 0$$

and then $x_{n-2}+t_{n-2} = t_{n-1}$. This contradicts the inequality $t_{n-1} < x_{n-2}+t_{n-2}$ since these are the lengths of the sides of the non degenerate triangle $(M_{n+1}, M_{n-2}, M_{n-1})$. The Jacobian matrix of the map $\gamma : \Omega_{n+1} \subset \mathbb{R}^{2n-1} \to \mathbb{R}^{n-2}$ is then of rank $n-2$. The proof by induction is then over.

We deduce that, for any $n \geq 4$, the map :

$$\gamma : \Omega_n \subset \mathbb{R}^{2n-3} \to \mathbb{R}^{n-3}$$

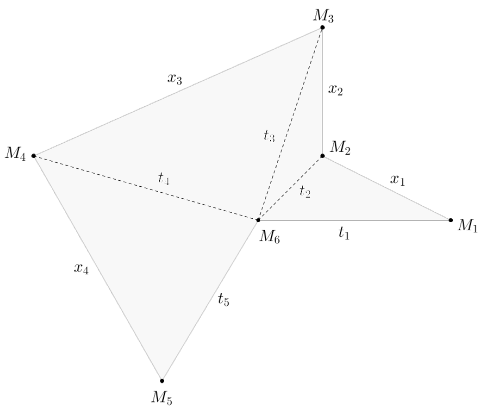
has maximal rank and then the nonempty set $\Gamma_n = \gamma^{-1}(\{0\})$ is a codimension $(n-3)$ submanifold of the open set Ω_n of $\mathbb{R}^{2n-3}$. This implies that Γ_n is a submanifold of dimension n of $\mathbb{R}^{2n-3}$. □

4 The area and perimeter foliations

In all this section, the space $\mathcal{P}_n^*$ of star-shaped polygons will be identified (as we have proceeded until now) to the open set Ω_n. [1] is a good reference for an elementary introduction to foliation theory.

We define the maps $p : \Omega_n \longrightarrow \mathbb{R}$, $\mathcal{A} : \Omega_n \longrightarrow \mathbb{R}$ and $\Psi : \Omega_n \longrightarrow \mathbb{R}^2$ by:

$$\begin{aligned} p(\omega) &= \text{perimeter of } \omega \\ \mathcal{A}(\omega) &= \text{area of } \omega \\ \Psi(\omega) &= (p(\omega), \mathcal{A}(\omega)). \end{aligned}$$



For any $\omega = (t_1, x_1, t_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n$, we have:

$$\begin{aligned} p(\omega) &= t_1 + x_1 + x_2 + \cdots + x_{n-2} + t_{n-1} \\ \mathcal{A}(\omega) &= \frac{1}{4}\sqrt{f(\omega_1)} + \cdots + \frac{1}{4}\sqrt{f(\omega_{n-2})} \end{aligned}$$

with:

$$\begin{aligned} &\omega_k = (t_k, x_k, t_{k+1}) \in \mathcal{V} \text{ for } k \in \{1, ..., n-2\} \\ &\text{area of } \omega_k = \frac{1}{4}\sqrt{f(\omega_k)} \text{ for } k \in \{1, \cdots, n-2\} \end{aligned}$$

where

$$f(x,y,z) = (x+y+z)(-x+y+z)(x-y+z)(x+y-z)$$

for $(x,y,z) \in \mathcal{V}$ (Héron's formula).

Setting $s(x,y,z) = x+y+z$, we obtain for any $v = (x,y,z) \in \mathcal{V}$:

$$f(v) = s(v)\,(s(v)-2x)\,(s(v)-2y)\,(s(v)-2z)\,.$$

On the other hand, the maps p, $\mathcal{A}$ and Ψ are clearly differentiable with gradient vectors $\nabla p(\omega)$ and $\nabla\mathcal{A}(\omega)$ given by:

$$\begin{aligned} &\nabla p(\omega) \\ &= \left(\frac{\partial p}{\partial t_1}(\omega), \frac{\partial p}{\partial x_1}(\omega), \cdots, \frac{\partial p}{\partial x_{n-2}}(\omega), \frac{\partial p}{\partial t_{n-1}}(\omega)\right) \\ &= (1,1,0,1,\cdots,0,1,1) \end{aligned}$$

and

$$\begin{aligned} &\nabla\mathcal{A}(\omega) \\ &= \left(\frac{\nabla f(\omega_1)}{8\sqrt{f(\omega_1)}}, \cdots, \frac{\nabla f(\omega_{n-2})}{8\sqrt{f(\omega_{n-2})}}\right) \\ &= \left(\frac{\sqrt{f(\omega_1)}}{8}\cdot\frac{\nabla f(\omega_1)}{f(\omega_1)}, \cdots, \frac{\sqrt{f(\omega_{n-2})}}{8}\cdot\frac{\nabla f(\omega_{n-2})}{f(\omega_{n-2})}\right) \end{aligned}$$

where the logarithmic derivative $\frac{\nabla f}{f}$ is given at each point $v = (x,y,z) \in \mathcal{V}$ by:

$$\begin{aligned} \frac{\nabla f(v)}{f(v)} &= \frac{\nabla s(v)}{s(v)} + \frac{\nabla(s(v)-2x)}{s(v)-2x} \\ &\quad + \frac{\nabla(s(v)-2y)}{s(v)-2y} + \frac{\nabla(s(v)-2z)}{s(v)-2z}. \end{aligned}$$

We then deduce:

$$\begin{aligned} \frac{1}{f(v)}\frac{\partial f}{\partial x}(v) &= \frac{1}{s(v)} - \frac{1}{s(v)-2x} \\ &\quad + \frac{1}{s(v)-2y} + \frac{1}{s(v)-2z} \\ \frac{1}{f(v)}\frac{\partial f}{\partial y}(v) &= \frac{1}{s(v)} + \frac{1}{s(v)-2x} \\ &\quad - \frac{1}{s(v)-2y} + \frac{1}{s(v)-2z} \\ \frac{1}{f(v)}\frac{\partial f}{\partial z}(v) &= \frac{1}{s(v)} + \frac{1}{s(v)-2x} \\ &\quad + \frac{1}{s(v)-2y} - \frac{1}{s(v)-2z}. \end{aligned}$$

This gives the partial derivatives of $\mathcal{A}$:

$$\text{for } k \in \{1,\cdots,n-2\},\ \frac{\partial \mathcal{A}}{\partial x_k}(\omega) = \frac{\sqrt{f(\omega_k)}}{8}\left(\frac{1}{s(\omega_k)} + \frac{1}{s(\omega_k)-2t_k} - \frac{1}{s(\omega_k)-2x_k} + \frac{1}{s(\omega_k)-2t_{k+1}}\right)$$

$$\frac{\partial \mathcal{A}}{\partial t_1}(\omega) = \frac{\sqrt{f(\omega_1)}}{8}\left(\frac{1}{s(\omega_1)} - \frac{1}{s(\omega_1)-2t_1} + \frac{1}{s(\omega_1)-2x_1} + \frac{1}{s(\omega_1)-2t_2}\right)$$

$$\text{for } k \in \{2,\cdots,n-2\},\ \frac{\partial \mathcal{A}}{\partial t_k}(\omega) = \frac{\sqrt{f(\omega_{k-1})}}{8}\left(\frac{1}{s(\omega_{k-1})} + \frac{1}{s(\omega_{k-1})-2t_{k-1}} + \frac{1}{s(\omega_{k-1})-2x_{k-1}} - \frac{1}{s(\omega_{k-1})-2t_k}\right) + \frac{\sqrt{f(\omega_k)}}{8}\left(\frac{1}{s(\omega_k)} + \frac{1}{s(\omega_k)-2x_k} + \frac{1}{s(\omega_k)-2t_{k+1}} - \frac{1}{s(\omega_k)-2t_k}\right)$$

$$\frac{\partial \mathcal{A}}{\partial t_{n-1}}(\omega) = \frac{\sqrt{f(\omega_{n-2})}}{8}\left(\frac{1}{s(\omega_{n-2})} + \frac{1}{s(\omega_{n-2})-2t_{n-2}} + \frac{1}{s(\omega_{n-2})-2x_{n-2}} - \frac{1}{s(\omega_{n-2})-2t_{n-1}}\right)$$

Theorem 4.1. *We have the following assertions.*
(1) The perimeter function p and the area function $\mathcal{A}$ are submersions on Ω_n. Then the level sets of p (resp. of $\mathcal{A}$) are leaves of a codimension 1 foliation $\mathcal{F}_p$ (resp. $\mathcal{F}_a$) on Ω_n.
(2) For $\omega \in \Omega_n$, the differential $d\Psi(\omega)$ is of rank 2 if ω is not a regular polygon and of rank 1 if ω is a regular polygon. Then the map Ψ defines a codimension 2 foliation $\mathcal{F}$ on the open set Ω_n^* of Ω_n consisting of non regular polygons.

Proof. Let $\omega = (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1})$ be an element of Ω_n and $(M_1, \cdots, M_n)$ one of its representatives as a star-shaped polygon.

Claim (1):
⋆ For any $\omega \in \Omega_n$, $dp(\omega) \neq 0$ since $\frac{\partial p}{\partial t_1}(\omega) = 1 \neq 0$. Then p is a submersion on Ω_n.
⋆ For any $\omega \in \Omega_n$, $\frac{\partial \mathcal{A}}{\partial t_1}(\omega) \neq 0$ or $\frac{\partial \mathcal{A}}{\partial x_1}(\omega) \neq 0$. Indeed we have the implications:

$$\frac{\partial \mathcal{A}}{\partial t_1}(\omega) = 0 = \frac{\partial \mathcal{A}}{\partial x_1}(\omega) \implies \frac{\partial \mathcal{A}}{\partial t_1}(\omega) + \frac{\partial \mathcal{A}}{\partial x_1}(\omega) = 0 \implies \frac{2}{s(\omega_1)} + \frac{2}{s(\omega_1)-2t_2} = 0 \implies t_1 + x_1 = 0.$$

But the equality $t_1 + x_1 = 0$ can not be satisfied. Then $d\mathcal{A}(\omega) \neq 0$. This proves that $\mathcal{A}$ is a submersion on Ω_n.

Claim (2): The Jacobian matrix $\mathcal{J}(\Psi, \omega)$ of the map Ψ at the point ω is the matrix

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 1 & 1 \\ \frac{\partial \mathcal{A}}{\partial t_1}(\omega) & \frac{\partial \mathcal{A}}{\partial x_1}(\omega) & \frac{\partial \mathcal{A}}{\partial t_2}(\omega) & \cdots & \frac{\partial \mathcal{A}}{\partial t_{n-2}}(\omega) & \frac{\partial \mathcal{A}}{\partial x_{n-2}}(\omega) & \frac{\partial \mathcal{A}}{\partial t_{n-1}}(\omega) \end{pmatrix}$$

consisting of the partial derivatives of the perimeter function on the first line and those of the area function on the second line. It is of rank 2 if one of the following conditions (C) is satisfied:
(i) *The partial derivatives $\frac{\partial \mathcal{A}}{\partial t_1}(\omega), \frac{\partial \mathcal{A}}{\partial t_{n-1}}(\omega), \frac{\partial \mathcal{A}}{\partial x_k}(\omega)$ are not all equal, for $k \in \{1, \cdots, n-2\}$;*

or

(ii) *at least one of the derivatives $\frac{\partial \mathcal{A}}{\partial t_k}(\omega)$ is not zero, for $k \in \{2, \cdots, n-2\}$.*

The function f appearing in the expressions of the partial derivatives is defined on $\mathcal{V}$ by:

$$f(x,y,z) = (x+y+z)(-x+y+z)(x-y+z)(x+y-z)$$

To simplify the calculations, we denote $(t_{k-1}, x_{k-1}, t_k, x_k, t_{k+1})$ simply (a, b, c, d, e); for $1 \leq k \leq n-1$ (agreeing that $t_0 = t_n = 0$ and $x_0 = t_1$) and for $(x,y,z) \in \mathcal{V}$, $g(x,y,z)$ will be the quantity:

$$\frac{1}{x+y+z} + \frac{1}{-x+y+z} + \frac{1}{x-y+z} - \frac{1}{x+y-z}$$

which is also:

$$g(x,y,z) = \frac{4z(x^2+y^2-z^2)}{f(x,y,z)}.$$

This is to prove that: *if ω is not a regular polygon, then the condition (C) is satisfied* ; or in an equivalent way: *if the condition (C) is not satisfied then ω is a regular polygon.*

We assume that the following condition $Non(C)$ is satisfied:

Non(C): *The partial derivatives* $\frac{\partial\mathcal{A}}{\partial t_1}(\omega)$, $\frac{\partial\mathcal{A}}{\partial t_{n-1}}(\omega)$, $\frac{\partial\mathcal{A}}{\partial x_k}(\omega)$ *are all equal for* k *in* $\{1,\cdots,n-2\}$ *and* $\frac{\partial\mathcal{A}}{\partial t_k}(\omega)$ *are all zero for* k *in* $\{2,\cdots,n-2\}$.

In this case, the matrix $\mathcal{J}(\Psi,\omega)$ is of rank 1.
Note that we have the following implications:

• **Implication (I):**

$$\begin{aligned}\text{Non(C)} &\Longrightarrow \frac{\partial\mathcal{A}}{\partial x_1}(\omega)-\frac{\partial\mathcal{A}}{\partial t_1}(\omega)=0\\ &\Longrightarrow \frac{\sqrt{f(c,d,e)}}{2}\cdot(g(e,c,d)-g(d,e,c))=0\\ &\Longrightarrow 2\sqrt{f(c,d,e)}\cdot\frac{c-d}{(e-c+d)(e+c-d)}=0\\ &\Longrightarrow c=d\\ &\Longrightarrow M_nM_1=M_1M_2\end{aligned}$$

• **Implication (II):**

$$\text{Non(C)} \Longrightarrow \text{for} k=2,\cdots n-2,\ \begin{cases}\frac{\partial\mathcal{A}}{\partial t_k}(\omega)=0\\ \text{and}\\ \frac{\partial\mathcal{A}}{\partial x_{k-1}}(\omega)=\frac{\partial\mathcal{A}}{\partial x_k}(\omega)\end{cases}$$

$$\begin{aligned}&\Longrightarrow \begin{cases}f(a,b,c)g(a,b,c)^2=f(c,d,e)g(d,e,c)^2\\ \text{and}\\ f(a,b,c)g(c,a,b)^2=f(c,d,e)g(e,c,d)^2\end{cases}\\ &\Longrightarrow f(a,b,c)\left(g(a,b,c)^2-g(c,a,b)^2\right)\\ &\quad=f(c,d,e)\left(g(d,e,c)^2-g(e,c,d)^2\right)\\ &\Longrightarrow (b^2-c^2)=(d^2-c^2)\\ &\Longrightarrow b=d\\ &\Longrightarrow M_{k-1}M_k=M_kM_{k+1}\end{aligned}$$

• **Implication (III):**

$$\begin{aligned}\text{Non(C)} &\Longrightarrow \frac{\partial\mathcal{A}}{\partial x_{n-2}}(\omega)-\frac{\partial\mathcal{A}}{\partial t_{n-1}}(\omega)=0\\ &\Longrightarrow \frac{\sqrt{f(a,b,c)}}{2}\cdot(g(c,a,b)-g(a,b,c))=0\\ &\Longrightarrow 2\sqrt{f(a,b,c)}\cdot\frac{c-b}{(a-b+c)(a+b-c)}=0\\ &\Longrightarrow b=c\\ &\Longrightarrow M_{n-2}M_{n-1}=M_{n-1}M_n\end{aligned}$$

From the three implications above, we deduce that ω is equilateral. Now, let us show that ω is regular.

For $k\in\{2,\cdots,n-2\}$, we have :

$$\frac{\partial\mathcal{A}}{\partial x_{k-1}}(\omega)=\frac{\partial\mathcal{A}}{\partial x_k}(\omega)$$

which implies:

$$\left(\frac{\partial\mathcal{A}}{\partial x_{k-1}}(\omega)\right)^2-\left(\frac{\partial\mathcal{A}}{\partial x_k}(\omega)\right)^2=0$$

Thus, taking into account the fact that the sides x_k are all equal in this case, we obtain by factorization:

$$(a-e)(ae-b^2+c^2)(ae+b^2-c^2)=0$$

• If $a-e=0$, taking into account the relation $\frac{\partial\mathcal{A}}{\partial t_k}(\omega)=0$, we obtain:

$$b^2+e^2=c^2=b^2+a^2$$

This implies that the two triangles (M_n,M_{k-1},M_k) and (M_n,M_{k+1},M_k) have right angles respectively at M_{k-1} and M_{k+1}. Consequently the quadrilateral $(M_n,M_{k-1},M_k,M_{k+1})$ is inscribable because:

$$\widehat{M_{k-1}}=\frac{\pi}{2}=\widehat{M_{k+1}}$$

• If $ae-b^2+c^2=0$, the quadrilateral $(M_n,M_{k-1},M_k,M_{k+1})$ has the following properties:

$$\begin{aligned}M_{k-1}M_k &= b=M_kM_{k+1}\ (\omega \text{ is equilateral})\\ c^2 &= b^2-ae\\ \cos\widehat{M_{k-1}} &= \frac{c^2-a^2-b^2}{2ab}\\ &= \frac{b^2-ae-a^2-b^2}{2ab}\\ &= -\frac{e+a}{2b}\\ \cos\widehat{M_{k+1}} &= \frac{c^2-e^2-b^2}{2eb}\\ &= \frac{b^2-ae-e^2-b^2}{2eb}\\ &= -\frac{a+e}{2b}\\ \cos\widehat{M_{k-1}} &= \cos\widehat{M_{k+1}}\end{aligned}$$

By the cosine's law of Al-Kashi, $a=e$ and, like in the previous case, the quadrilateral $(M_n,M_{k-1},M_k,M_{k+1})$ is inscribable.

• If $ae+b^2-c^2=0$, the quadrilateral $(M_n,M_{k-1},M_k,M_{k+1})$ is still inscribable since it satisfies the relation $\cos\widehat{M_{k-1}}=-\cos\widehat{M_{k+1}}$. Indeed, we have:

$$\begin{aligned}c^2 &= ae+b^2\\ \cos\widehat{M_{k-1}} &= \frac{c^2-a^2-b^2}{2ab}\\ &= \frac{b^2+ae-a^2-b^2}{2ab}\\ &= \frac{e-a}{2b}\\ \cos\widehat{M_{k+1}} &= \frac{c^2-e^2-b^2}{2eb}\\ &= \frac{b^2+ae-e^2-b^2}{2eb}\\ &= \frac{a-e}{2b}\end{aligned}$$

We have proved in all cases that for any k in $\{2, \cdots, n-2\}$, the quadrilateral $(M_n, M_{k-1}, M_k, M_{k+1})$ is inscribable. This implies that the polygon $(M_1, \cdots, M_n)$ is inscribable. Since the latter is equilateral, it is necessarily regular.

Finally the singular points of the map Ψ are the regular polygons and this map induces a submersion on Ω_n^* whose level sets are leaves of a foliation $\mathcal{F}$. □

5 The example of triangles

This is the situation where we see things very concretely and where drawings can be made explicit. The way we treat the topic in this section will be slightly different from the previous one.

5.1 The space of non degenerate triangles

To give oneself a non degenerate triangle in a Euclidean finite dimensional space is to give oneself three ordered real numbers $x > 0$, $y > 0$ and $z > 0$ such that

$$x < y + z$$
$$y < z + x$$
$$z < x + y$$

which represent the lengths of the sides. Exceptionally in this section, we shall denote a triangle by $\langle xyz \rangle$ instead of (X, Y, Z) where the points X, Y and Z are the vertices. Indeed it is well known that $\langle xyz \rangle$ is isometric to $\langle x'y'z' \rangle$ if :

$$x = x', \quad y = y', \quad z = z'.$$

(For the moment we will make the difference between a triangle and another obtained by permutation of the three numbers representing it even if, geometrically, they are the same!) From now on, λ will be the half perimeter $\lambda = \frac{x+y+z}{2}$.

The set of non degenerate triangles is thus the open set $\Omega_3 \subset (\mathbb{R}_+^*)^3$ given by (1.7). We will describe it explicitly. To inequalities (5.1) are associated three equations defining respectively three planes:

$$\begin{aligned} \Sigma_1 &= x = y + z \\ \Sigma_2 &= \{y = z + x\} \\ \Sigma_3 &= \{z = x + y\} \end{aligned} \tag{5.1}$$

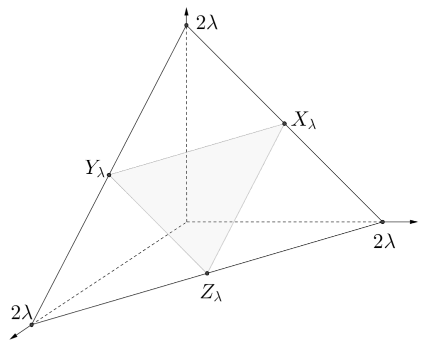
In the slice $\{x + y + z = 2\lambda\}$ of $\mathbb{R}_+^3$, Σ_1, Σ_2 and Σ_3 are the sides of an equilateral triangle in $\mathbb{R}_+^3$ whose vertices are :

$$X_\lambda = (0, \lambda, \lambda), \quad Y_\lambda = (\lambda, 0, \lambda), \quad Z_\lambda = (\lambda, \lambda, 0)$$

(see the picture bellow); the interior P_λ of the convex hull of these three points represents the space of triangles $\langle xyz \rangle$ whose perimeter is 2λ.

When λ varies to λ', we obtain another $P_{\lambda'}$, image of P_λ by the homothety centered at the origin and with ratio $k = \frac{\lambda'}{\lambda}$. Thus, the space Ω_3 is *foliated* by these P_λ; Ω_3 is in fact the open cone with vertex the origin and basis anyone of these leaves P_λ, for instance P_1:

$$\Omega_3 = \bigcup_{\lambda \in \mathbb{R}_+^*} \lambda P_1 = \{\lambda X : \ X \in P_1 \text{ and } \lambda \in \mathbb{R}_+^*\}$$



For a particular situation which will appear thereafter, we recall the following result stated in Theorem 6.1 in the general case of polygons.

> *For a given family of triangles with prescribed perimeter, the maximum of the area is realized by the equilateral triangle.*

5.2 The perimeter foliation $\mathcal{F}_p$

Each P_λ (where $\lambda \in \mathbb{R}_+^*$) is the level set $p(x, y, z) = 2\lambda$ where p is the perimeter function $p(x, y, z) = x + y + z$. We have also seen that the level surface P_λ is the interior of the convex hull of the triangle $X_\lambda Y_\lambda Z_\lambda$.

Thus we have a foliation $\mathcal{F}_p$ on Ω_3 whose leaves are the surfaces P_λ $(\lambda > 0)$. Of course, $\mathcal{F}_p$ is trivial since isomorphic to the product $P_1 \times \mathbb{R}_+^*$.

5.3 The area foliation $\mathcal{F}_a$

The function $\mathcal{A} : (\mathbb{R}_+^*)^3 \longrightarrow \mathbb{R}_+^*$ which associates to a triangle $\langle xyz \rangle$ its area is given by Héron formula:

$$\mathcal{A}(x, y, z) = \frac{1}{4}\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}$$

The foliation $\mathcal{F}_a$ in which we will be interested is the foliation whose leaves are the level surfaces of this function.

• The surface at level s of the function $\mathcal{A}$ on the open set Ω_3 is exactly the surface at level s^2 of the function $\Phi = \mathcal{A}^2$. The benefit of working with Φ instead of $\mathcal{A}$ is that there is no more square root, which simplifies the calculations, among others that of the differential which plays a fundamental role. We consider then the function:

$$\Phi(x, y, z) = \frac{1}{16}(x+y+z)(-x+y+z)(x-y+z)(x+y-z)$$

• The differential of Φ has the form:

$$d\Phi(x,y,z) = \frac{1}{16}\{A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz\}$$

where the functions A, B and C are given as follows:

$$\begin{aligned} A &= -x+y+z)(x-y+z)(x+y-z) \\ &\quad -(x+y+z)(x-y+z)(x+y-z) \\ &\quad +(x+y+z)(-x+y+z)(x+y-z) \\ &\quad +(x+y+z)(-x+y+z)(x-y+z) \\ B &= (-x+y+z)(x-y+z)(x+y-z) \\ &\quad +(x+y+z)(x-y+z)(x+y-z) \\ &\quad -(x+y+z)(-x+y+z)(x+y-z) \\ &\quad +(x+y+z)(-x+y+z)(x-y+z) \\ C &= (-x+y+z)(x-y+z)(x+y-z) \\ &\quad +(x+y+z)(x-y+z)(x+y-z) \\ &\quad +(x+y+z)(-x+y+z)(x+y-z) \\ &\quad -(x+y+z)(-x+y+z)(x-y+z) \end{aligned} \quad (5.2)$$

An easy but long computation shows that these three functions A, B et C are zero simultaneously only if $x = y = z = 0$, which can not happen since $(0,0,0)$ is not in Ω_3.

• If we fix the perimeter 2λ, the area function a is maximal, and so is the function Φ, when $x = y = z = \frac{2}{3}\lambda$; at this point Φ is equal to $\frac{\lambda^4}{27}$. These are the values taken by the function Φ on the open half line Δ whose equation is $x = y = z$.

Now let Ω_3^* be the open set $\Omega_3 \setminus \Delta$. At $u = (x,y,z) \in \Omega_3^*$, the differential $d_u\Phi$ has rank 1; then the set level of Φ passing through this point is a regular surface A, in fact an algebraic surface of degree 4. Its equation is:

$$(x+y+z)(-x+y+z)(x-y+z)(x+y-z) = 16\Phi(u).$$

Let G be the subgroup of $\mathrm{Isom}(\mathbb{R}^3)$ (the full group of isometries of the Euclidean space $\mathbb{R}^3$) generated by the rotation whose axis is Δ and angle $\frac{2\pi}{3}$ and the reflection σ with respect to the plane of equation $x = y$. (The restrictions of these elements to the plane of equation $x+y+z = 2\lambda$ is the group of isometries of the equilateral triangle $X_\lambda Y_\lambda Z_\lambda$.) It leaves the space Ω_3 invariant and also its boundary $\partial\Omega_3$, the half line Δ and the open sets Ω_3 and Ω_3^*. Then it acts on Ω_3 and fixes each leaf of $\mathcal{F}_a$; the same applies to the foliation $\mathcal{F}_p$.

5.4 The area-perimeter foliation

Let $\Psi : \Omega_3^* \longmapsto (\mathbb{R}_+^*)^2$ be the function:

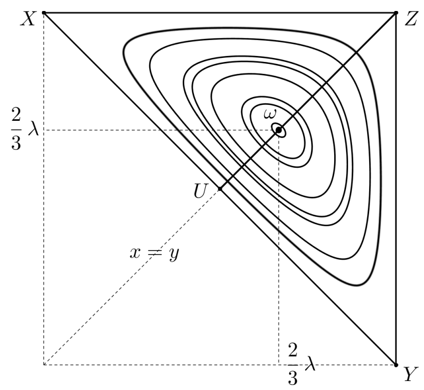
$$\Psi(x,y,z) = (p(x,y,z), \Phi(x,y,z)).$$

Up to a multiplicative factor, the matrix of its differential at $u = (x,y,z)$ is:

$$d_u\Psi = \begin{pmatrix} 1 & 1 & 1 \\ A(u) & B(u) & C(u) \end{pmatrix}$$

where A, B and C are the functions given by (5.2). It can be shown that these functions are equal only if $x = y = z$; then, for $u \in \Omega_3^*$, $d_u\Psi$ has rank 2. Thus, the level sets of Ψ are regular curves, leaves of a foliation $\mathcal{F}_*$ on Ω_3^*.

On Ω_3 we have a singular foliation $\mathcal{F} = \mathcal{F}_p \cap \mathcal{F}_a$. Its leaves of dimension 0 are the points of the open half line $\{\left(\frac{2}{3}\lambda, \frac{2}{3}\lambda, \frac{2}{3}\lambda\right) : \lambda \in \mathbb{R}_+\}$. The other leaves are of dimension 1; each one has equation $\Psi(u) = $ constant in the open set Ω_3^*. These curves define (by restriction) a foliation on each P_λ (leaf of $\mathcal{F}_p$). To see what it is, this P_λ is projected orthogonally on the plane $z = 0$; we obtain the foliation depicted in the figure below (drawn by GeoGebra). We will explain what all this means.



The interior of the triangle XYZ is the projection (which we denote by Θ_λ) on the plane $z = 0$ of the set P_λ of triangles $\langle x_\lambda y_\lambda z_\lambda \rangle$ with perimeter 2λ. Note that the boundary of P_λ is an equilateral triangle while XYZ is an isosceles and right triangle. The foliation $\mathcal{F}$ on P_λ is isomorphic to the foliation on the picture via the diffeomorphism $f : P_\lambda \longrightarrow \Theta_\lambda$ defined by $f(x,y,z) = (x,y,0)$ with inverse $f^{-1}(x,y,0) = (x,y,2\lambda - x - y)$.

• The point ω with coordinates $\left(\frac{2}{3}\lambda, \frac{2}{3}\lambda\right)$ corresponds to the equilateral triangle $\langle x_\lambda x_\lambda x_\lambda \rangle$ with maximal area. As we easily imagine, an equilateral triangle may never be deformed to an other one having the same perimeter and the same area.

• The curves at the interior of Θ_λ are leaves of a foliation of $\Theta_\lambda \setminus \{\omega\}$, each leaf corresponds to the set of triangles having the same area. It has:

$$\lambda(2\lambda - x)(2\lambda - y)(x+y) = 8c$$

as equation where c is a constant varying in the interval $\left]0, \frac{8\lambda^4}{27}\right[$.

• The piece UZ of diagonal corresponds to isosceles triangles (for which $x = y$). In each leaf, there is exactly the projections of two isosceles triangles $\langle xxz \rangle$ and $\langle x'x'z' \rangle$.

• One can see from the figure that the entire situation is invariant by the reflection σ (symmetry with respect to the diagonal $x = y$) while that on the triangle P_λ is invariant by the full group G. $\diamondsuit$

Example 5.1. Geoffrey Letellier constructed two lines of isosceles triangles: $\langle x_\lambda y_\lambda z_\lambda \rangle$ and $\langle x'_\lambda y'_\lambda z'_\lambda \rangle$ where $\lambda \in \mathbb{R}_+^*$ with $x_\lambda = y_\lambda = \frac{11}{14}\lambda$, $z_\lambda = \frac{3}{7}\lambda$ and $x'_\lambda = y'_\lambda = \frac{4}{7}\lambda$, $z'_\lambda = \frac{6}{7}\lambda$. They are such that, for any $\lambda \in \mathbb{R}_+^*$:

- $\langle x_\lambda x_\lambda z_\lambda \rangle$ and $\langle x'_\lambda x'_\lambda z'_\lambda \rangle$ have the same perimeter 2λ.
- $\langle x_\lambda x_\lambda z_\lambda \rangle$ and $\langle x'_\lambda x'_\lambda z'_\lambda \rangle$ have the same area $\frac{3\lambda^2}{7\sqrt{7}}$.
- $\langle x_\lambda x_\lambda z_\lambda \rangle$ and $\langle x'_\lambda x'_\lambda z'_\lambda \rangle$ are not isometric.

For instance, the two isosceles triangles $x = y = 11$, $z = 6$ and $x' = y' = 8$, $z' = 12$ have the same perimeter equal to 28 and the same area equal to $12\sqrt{7}$.

6 Results related to perimeter, area and convexity

The following well known classical results are among the most beautiful theorems that we can cite in Euclidean elementary geometry of the plane. (For a sketch of proof, see for instance [3].)

Theorem 6.1. *(Isoperimetric inequality). Among all the convex polygons with prescribed perimeter, the regular polygon is the one whose area is maximum.*

In [8] it is proved that, for given positive numbers $a_1, \cdots, a_n$ such that, for each $i = 1, \cdots, n$, $a_i \leq \sum_{k \neq i} a_k$, there exists a unique (up to an isometry) convex inscribable polygon with edge lengths $a_1, \cdots, a_n$.

In the same order of ideas, we also have the following theorem. Its proof is not difficult but it is a bit long and not immediate. (And the reader can even attempt to reproduce it himself!)

Theorem 6.2. *Among all convex polygons whose sides have given lengths, the inscribable polygon is the one whose area is maximum.*

Using the analytic expression of the function "area" $\mathcal{A} : \omega \in \Omega_n \longrightarrow \text{area}(\omega) \in \mathbb{R}$, we prove the following result related to the two theorems above. (It was also partially established, by a different method, in [5].)

Theorem 6.3. *We have the following assertions:*

(1) For any real number $L > 0$, the differentiable manifold $p^{-1}(\{L\})$ consisting of all star-shaped polygons with perimeter L, is diffeomorphic to a open set $\Omega_{n,L}$ of $\mathbb{R}^{2n-4}$ and the restriction $\mathcal{A}_L : \Omega_{n,L} \to \mathbb{R}$ of $\mathcal{A}$ to $\Omega_{n,L}$ admits a critical point at the unique regular polygon ω_L of perimeter L.

(2) The convex polygons whose sides have given lengths form a differentiable manifold diffeomorphic to an open set of $\mathbb{R}^{n-3}$ and the restriction of the function $\mathcal{A}$ to this open set admits a critical point at its unique inscribable polygon.

Proof. Recall that, for any

$$\omega = (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n,$$

we have:

$$p(\omega) = \text{perimeter}(\omega) = t_1 + x_1 + \cdots + x_{n-2} + t_{n-1}$$

$$\mathcal{A}(\omega) = \text{area}(\omega) = \frac{1}{4}\sqrt{f(\omega_1)} + \cdots + \frac{1}{4}\sqrt{f(\omega_{n-2})}$$

with $\omega_k = (t_k, x_k, t_{k+1})$ for $k \in \{1, \cdots, n-2\}$, and

$$\begin{aligned} & f(x, y, z) \\ = \ & (x + y + z)(-x + y + z)(x - y + z)(x + y - z) \end{aligned}$$

for $(x, y, z) \in \mathcal{V}$. Setting $h(v) = \frac{1}{4}\sqrt{f(v)}$ for $v \in \mathcal{V}$, we obtain:

$$\mathcal{A}(\omega) = h(\omega_1) + \cdots + h(\omega_{n-2})$$

for any $\omega = (t_1, x_1, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n$.

<u>Claim</u> (1): Let $L \in]0, +\infty[$.
For any $\omega = (t_1, x_1, t_2, ..., t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_n$, we have:

$$p(\omega) = L \Leftrightarrow t_1 = L - x_1 - \cdots - x_{n-2} - t_{n-1}$$

Then, by considering the affine map

$$T : \mathbb{R}^{2n-4} \to \mathbb{R}^{2n-3} = \mathbb{R} \times \mathbb{R}^{2n-4}$$

given, for $u = (x_1, t_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1})$, by:

$$T(u) = (t_1(u), u)$$

$$\text{where } \ t_1(u) = L - x_1 - \cdots - x_{n-2} - t_{n-1}$$

We see that $p^{-1}(\{L\})$ is naturally identified to the open set $\Omega_{n,L} = T^{-1}(\Omega_n)$ of $\mathbb{R}^{2n-4}$.

For $u = (x_1, t_2, \cdots, t_{n-2}, x_{n-2}, t_{n-1}) \in \Omega_{n,L}$, we have:

$$\begin{aligned} \mathcal{A}_L(u) \ &= h(t_1(u), x_1, t_2) + h(t_2, x_2, t_3) + \cdots \\ &\quad + h(t_{n-2}, x_{n-2}, t_{n-1}) \end{aligned}$$

Set $t_1 = t_1(u)$ and $\omega_k = (t_k, x_k, t_{k+1})$ for $k \leq n-2$. The partial derivatives for any $(x, y, z) \in \mathcal{V}$ are :

$$\begin{aligned} & \frac{\partial h}{\partial x}(x, y, z) \\ = \ & \frac{x\left(-x^2+y^2+z^2\right)}{2\sqrt{(x+y+z))(-x+y+z)(x-y+z)(x+y-z)}} \end{aligned}$$

$$\begin{aligned} & \frac{\partial h}{\partial y}(x, y, z) \\ = \ & \frac{y\left(x^2-y^2+z^2\right)}{2\sqrt{(x+y+z))(-x+y+z)(x-y+z)(x+y-z)}} \end{aligned}$$

$\frac{\partial h}{\partial z}(x,y,z)$
$= \frac{z\left(x^2+y^2-z^2\right)}{2\sqrt{(x+y+z))(-x+y+z)(x-y+z)(x+y-z)}}$

On the other hand

$\frac{\partial \mathcal{A}_L}{\partial x_1}(u)$
$= \frac{\partial}{\partial x_1}\left[h\left(t_1(u), x_1, t_2\right)\right]$
$= -\frac{\partial h}{\partial x}(\omega_1) + \frac{\partial h}{\partial y}(\omega_1)$
$= \frac{(t_1-x_1)(t_1+x_1-t_2)(t_1+x_1+t_2)}{2\sqrt{(t_1+x_1+t_2)(-t_1+x_1+t_2)(t_1-x_1+t_2)(t_1+x_1-t_2)}}$

$\frac{\partial \mathcal{A}_L}{\partial t_{n-1}}(u)$
$= \frac{\partial}{\partial t_{n-1}}\left[h\left(t_1(u), x_1, t_2\right) + h\left(t_{n-2}, x_{n-2}, t_{n-1}\right)\right]$
$= -\frac{\partial h}{\partial x}(\omega_1) + \frac{\partial h}{\partial z}(\omega_{n-2})$

Finally and for $k \in \{2, \cdots, n-2\}$:
$\frac{\partial \mathcal{A}_L}{\partial t_k}(u)$
$= \frac{\partial}{\partial t_k}\left[h\left(t_{k-1}, x_{k-1}, t_k\right) + h\left(t_k, x_k, t_{k+1}\right)\right]$
$= \frac{\partial h}{\partial z}(\omega_{k-1}) + \frac{\partial h}{\partial x}(\omega_k)$

$\frac{\partial \mathcal{A}_L}{\partial x_k}(u)$
$= \frac{\partial}{\partial x_k}\left[h\left(t_1(u), x_1, t_2\right) + h\left(t_k, x_k, t_{k+1}\right)\right]$
$= -\frac{\partial h}{\partial x}(\omega_1) + \frac{\partial h}{\partial y}(\omega_k)$

If the sides of the polygon are not all equal, there are two consecutive ones with a common vertex M and different lengths.

By changing the numbering of the vertices, one can assume $M = M_1$. In these conditions, the lengths $t_1 = M_nM_1$ and $x_1 = M_1M_2$ are different and this implies $\frac{\partial \mathcal{A}_L}{\partial x_1}(u) \neq 0$ and that u is not a critical point of $\mathcal{A}_L$.

If the lengths of all the sides are equal, then $t_1 = x_1 = x_2 = ... = x_{n-2} = t_{n-1}$ and a necessary condition for this polygon to be a critical point of $\mathcal{A}_L$, is $\frac{\partial \mathcal{A}_L}{\partial t_k}(u) = 0$ for any $k \in \{2, \cdots, n-2\}$, that is

$$\frac{\partial h}{\partial z}(t_{k-1}, x_k, t_k) = -\frac{\partial h}{\partial x}(t_k, x_k, t_{k+1})$$

for any $k \in \{2, \cdots, n-2\}$. This implies:

$$\left(\frac{\partial h}{\partial z}(t_{k-1}, x_k, t_k)\right)^2 - \left(\frac{\partial h}{\partial x}(t_k, x_k, t_{k+1})\right)^2 = 0$$

again for any $k \in \{2, \cdots, n-2\}$. By factorization, we have:

$$\frac{\Delta}{f(\omega_{k-1})f(\omega_k)} = 0$$

where:

$$\begin{aligned}\Delta \;=\; & (t_{k+1}t_{k-1} + x_k^2 - t_k^2)(t_{k+1}t_{k-1} - x_k^2 + t_k^2) \\ & (t_{k+1} + t_{k-1})(t_{k+1} - t_{k-1})x_k^2 t_k^2\end{aligned}$$

Thus

$$(t_{k+1} - t_{k-1})(t_{k+1}t_{k-1} - x_k^2 + t_k^2)(t_{k+1}t_{k-1} + x_k^2 - t_k^2) = 0$$

The proof ends like that of Theorem 4.1. We thus obtain the inscriptibility of all the quadrilaterals $(M_n, M_{k-1}, M_k, M_{k+1})$ for $k \in \{2, \cdots, n-2\}$ and then the inscriptibility of the polygon $(M_1, \cdots, M_n)$. But since the latter has all its sides of the same length, it is necessarily regular.

Finally the singular points of the map $\mathcal{A}_L$ are the regular polygons of $\Omega_{n,L}$, that is, the unique regular polygon ω_L of perimeter L.

Claim(2) : Let $v = (\bar{t}_1, \bar{x}_1, \cdots, \bar{x}_{n-2}, \bar{t}_{n-1}) \in \mathcal{V}_n$ and let F_v be the set of convex polygons whose sides are $\bar{t}_1$, $\bar{x}_1$,$\cdots$,$\bar{x}_{n-2}$, $\bar{t}_{n-1}$. This is an open set is of $\mathbb{R}^{n-3}$.

On the other hand, the area function $F_v \overset{\mathcal{A}}{\to} \mathbb{R}$ is given, for $t = (t_2, t_3, \cdots, t_{n-2}) \in F_v$, by:

$$\begin{aligned}\mathcal{A}(t) \;&= h(\bar{t}_1, \bar{x}_1, t_2) + h(t_2, \bar{x}_2, t_3) \\ &\quad + \cdots + h(t_{n-2}, \bar{x}_{n-2}, \bar{t}_{n-1})\end{aligned}$$

Setting:

$$\begin{aligned}\omega_1 &= (\bar{t}_1, \bar{x}_1, t_2) \\ \omega_k &= (t_k, \bar{x}_k, t_{k+1}) \text{ for} k \in \{2, \cdots, n-3\} \\ \omega_{n-2} &= (t_{n-2}, \bar{x}_{n-2}, \bar{t}_{n-1})\end{aligned}$$

one can express the partial derivatives of $\mathcal{A}$ as follows:

$$\begin{aligned}\frac{\partial \mathcal{A}}{\partial t_k}(t) &= \frac{\partial}{\partial t_k}\left[h\left(t_{k-1}, \bar{x}_{k-1}, t_k\right) + h\left(t_k, \bar{x}_k, t_{k+1}\right)\right] \\ &= \frac{\partial h}{\partial z}(\omega_{k-1}) + \frac{\partial h}{\partial x}(\omega_k)\end{aligned}$$

A critical point $t \in F_v$ of the area function must satisfy $\frac{\partial h}{\partial z}(\omega_{k-1}) + \frac{\partial h}{\partial x}(\omega_k) = 0$, and then

$$\left(\frac{\partial h}{\partial z}(\omega_{k-1})\right)^2 - \left(\frac{\partial h}{\partial x}(\omega_k)\right)^2 = 0.$$

Setting $\omega_{k-1} = (u, v, w)$ and $\omega_k = (w, s, r)$, we obtain $\frac{\alpha}{\beta} = 0$ where:

$$\begin{aligned}\alpha = \; & w^2\left(r^2uv + rsu^2 + rsv^2 - rsw^2 + s^2uv - uvw^2\right) \\ & \left(-r^2uv + rsu^2 + rsv^2 - rsw^2 - s^2uv + uvw^2\right).\end{aligned}$$

and $\beta = f(u,v,w).f(w,s,r)$, or

$$[(rsu^2 + rsv^2 - rsw^2) + (s^2uv - uvw^2 + r^2uv)]$$
$$[(rsu^2 + rsv^2 - rsw^2) - (s^2uv - uvw^2 + r^2uv)] = 0.$$

This implies:

$$\Big(rsu^2 + rsv^2 - rsw^2\Big)^2 - \Big(s^2uv - uvw^2 + r^2uv)\Big)^2 = 0$$

or

$$rs(u^2 + v^2 - w^2) = \pm uv(r^2 + s^2 - w^2)$$

Thus we deduce:

$$\cos \widehat{M_{k-1}} = \pm \cos \widehat{M_{k+1}}$$

which implies that the quadrilateral $(M_n, M_{k-1}, M_k, M_{k+1})$ is inscribable for any index $k \in \{2, \cdots, n-2\}$. Hence the polygon t is inscribable.

Finally, if $t \in F_v$ is inscribable then, one can prove easily that t is a critical point of the area function. This completes the proof of Theorem 6.3. □

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