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Open question in dynamical systems: the rolling problem

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Abstract

In this paper, we present one of the simplest dynamical systems, inspired by Contact geometry and topology. It is a problem of double tangencies to a real curve in the plane. The solution to the problems stated in this article requires a good understanding of probability theory in the spirit of percolation theory in mathematical physics. A complete solution to the open problems presented here would be a major breakthrough in mathematics.

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1 The Problem

We describe here an open problem of dynamics in contact geometry, that also leads to a problem in algebraic geometry. Let's start with what could be considered as the simplest problem of geometry.

Let $f : S^1 \to R^2$ be a smooth (C^∞) curve from the oriented unit circle to the real plane. Assume the following genericity conditions:

(i) its derivative does not vanish (i.e this is an immersion)

(ii) when the curvature vanishes, its derivative does not (i.e the inflexion points are isolated)

(iii) no line of the plane is tangent to more than two points of the curve.

Then consider the FINITE set S of all lines in the plane that are tangent to the curve at two points (prove that S is finite). Now consider the set $\tilde{S}$ made of two copies of each tangent line in S, one with one orientation and the other with the opposite orientation. Thus the cardinality of $\tilde{S}$ is twice that of S. This yields the simplest possible dynamical system in the following way: start at any point p_0 of the curve, consider the tangent line L_{p_0} to f at p_0 and the one-parameter family L_{p_t} of tangent lines as the point p_t runs on the curve in the positive direction, until one reaches a point p_1 of the curve where the tangent to the curve at that point belongs to S (that is to say the tangent at this point p_1 is also tangent to another point p_2 of the curve). Then jump from p_1 to p_2 and continue from p_2 in the positive direction of the curve until one reaches another line of S tangent to p_2 and p_3 and then jump to p_3, and so on. This yields a dynamical system on the oriented tangent lines of $\tilde{S}$. Indeed, as the starting point p_0 varies on all of the image of f, this algorith partitions this set $\tilde{S}$ into a finite number of cycles. Indeed, every double tangent line L will occur exactly twice: if say L is tangent to the two points x_0, x_1 of the curve, the jump on L from x_0 to x_1 will appear once, and the reverse jump on L from x_1 to x_0 will appear once. See Figure 1.

Normalize the curve so that its length be 1 and assume that it is parametrized by arc length. Define the energy E of the curve f as the integral of the square of its curvature. Here are the questions:

Problem 1: What can be said of this dynamical system when the energy becomes high ? How does the cardinality of S increases as a (probabilistic) function of E ? A conjecture is the following: for any p and q, both in $[0, 1]$ and as close to 1 as one wishes, there is an energy $E(p, q)$ large enough so that any curve of energy larger or equal to $E(p, q)$ has probability p of containing one big q-cycle. By big q-cycle, we mean a cycle of $\tilde{S}$ that contains at least q times the cardinality of $\tilde{S}$. Intuitively, this means that as the energy increases, there is almost always only one big cycle, the others being small.

Problem 2: For any energy and especially for high energies, the curve will go round in the clockwise direction n_1 times (positive curvature), and then reach an inflexion point, and then go round n_2 times in the anti-clockwise direction (negative curvature), and so on... If there is no cost in changing the sign of the curvature,

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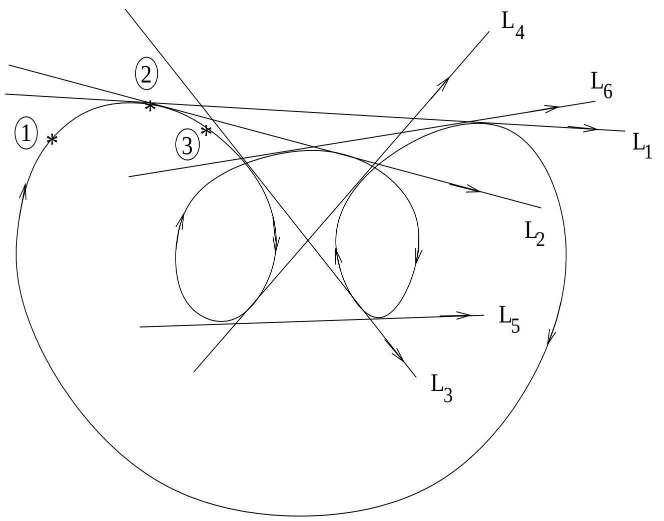

Fig. 1: $S = \{L_1, L_2, L_3, L_4, L_5, L_6\}$
$\widetilde{S} = \{L_1, -L_1, L_2, -L_2, \ldots, L_6, -L_6\}$
Note: The orientations of L_i's are chosen arbitrarily.
With starting point ①, we get the cycle $\{L_1\}$.
With starting point ②, we get the cycle $\{L_2, -L_5, L_6, -L_1\}$.
With starting point ③, we get the cycle $\{L_3, -L_4, L_5, -L_3, L_4, -L_6, -L_2\}$.
The union of the 3 cycles gives $\widetilde{S}$.
Any other starting point will lead to one of these 3 cycles.

the sequence n_i will be the one that one would obtain by throwing each time a coin. Is this close to a brownian motion in the limit of high energies ?

Problem 3: Percolation: assuming that S^1 is replaced by the half real line $[0, \infty)$, with starting point at the center of a given circle, say of radius one, what is the probability of reaching the circle in time t given the energy E of the curve ? Is this the same as a brownian motion when the energy tends to infinity ?

Problem 4: What is the right measure to put on maps f ? Once developed in Fourier series by positive and negative exponents of the exponential (by viewing R^2 as the complex plane), one could either take the gaussian measure for the coefficients, or a double Poisson measure (one on the positive rotations, one on the negative rotations).

1.1 Contact Geometry and Generalizations

Consider the space $J^1(R^n, R)$ of 1-jets of maps from R^n to R. Take the coordinates x for the independent coordinate, z for the function of x, and y for the first derivative of z with respect to x. Consider the contact form $dz - \Sigma y_i dx_i$. Then the Reeb vector field is $\frac{\partial}{\partial z}$ and the contact distribution ξ is generated by the vectors

$$\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z},\ 1 \leq i \leq n.$$

Now, instead of this standard Reeb vector field, take the following vector field: $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$. This choice is not intrinsic from the point of view of contact topology. However, it is intrinsic from the point of view of contact geometry. Observe that X is included in the contact hyperplane distribution, and its slope is simply y itself. Now if L is a Legendrian curve in $J^1(R, R) = R^3$, with this standard choice of coordinates and contact form, a X-chord between two points of L, assuming that L is the 1-jet of a map $f : R \to R$, is a line that is tangent to the graph of f at two points ! And we recover the problem of double tangencies above. It is interesting to note that, even if X is included in the contact distribution like L, they are never colinear except exceptionally when the Legendrian curve L is tangent to a hyperplane $y = const$ in which case the graph of the map $z(x)$ has an inflection point.

Problem 5: What are the geometric invariants in this problem ? Generalize this setting to tangent planes of higher dimensional manifolds. Say, for a surface in R^3, the tangent plane would roll on two points until it reaches a tangent plane with three points of tangency.

Problem 6: Given a smooth projective submanifold M of CP^n of complex dimension k, compute the dimension of the projective submanifold $\Phi(M)$ made of all lines in CP^n that are tangent to two points of M. Show that for a submanifold of complex dimension 3 in CP^5, the resulting manifold has the same dimension 3. Open question: how to determine if a projective manifold lies in the image of the functor Φ ?

Remark: I thank Egor Shelukhin for telling me that a discretised version of this problem appeared as a question in the World Mathematics Olympiads. The question there was to prove that, given any finite set of points in the real plane, where no triple of points are colinear, such a rolling algorithm where the line turns always clockwise, covers in a single cycle all points of the set if the starting point is well-chosen.

There is no reference for the problem itself, but here is a general reference to contact geometry [1]. Note that this problem is fundamentally a problem of probability theory.

References

[1] H. Geiges, *Contact Geometry*, Handbook of Differential Geometry vol. 2 (F.J.E. Dillen and L.C.A. Verstraelen, eds.), North-Holland, Amsterdam (2006), pp. 315-382

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