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The fundamental group of the punctured Klein bottle and the simple loop conjecture

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Abstract

In this paper we provide a classification of fundamental group elements representing simple closed curves on the punctured Klein bottle, Similar to the Birman-Series classification of curves on the punctured torus [1]. In the process, an explicit description of the mapping class group is given. We then apply this to give a counterexample to the simple loop conjecture for representations from the Klein bottle group to $PGL(2, \mathbb{R})$.

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1 Introduction

The modern study of surfaces (i.e. two-dimensional manifolds) frequently exploits the interplay between two objects rich in structure: their mapping class groups, on the one hand, and on the other their collections of simple closed curves. Here by a *simple closed curve* in a surface F we mean a subspace of F homeomorphic to S^1. Simple closed curves are thus codimension-one subspaces that intersect transversely in finite sets, so one can extract information about surfaces and their associated objects by cutting them into pieces along simple closed curves or by studying the intersection patterns of their simple closed curves. In particular, W.J. Harvey defined a *complex of curves* for a topological surface F, a simplicial complex with a vertex for each isotopy class of non-nullhomotopic simple closed curves of F and an n-simplex for each collection of $n+1$ curves that can be isotoped pairwise disjoint [9]. An extensive body of literature has since accumulated that describes and uses the structure of this complex, see for instance the seminal works of Harer [8] and Masur–Minsky [19], [18].

It is easy to see with a little scratch work that the complex of curves is very complicated (eg. locally infinite) even for relatively simple surfaces. But for surfaces of very low topological complexity it is still possible to explicitly understand the entire collection of simple closed curves, and it can be quite useful to take several perspectives on this. For instance, simple closed curves on the torus $S^1 \times S^1$ are well known to correspond bijectively to $\mathbb{Q} \cup \{\infty\}$, and the complex of curves (here redefined so that vertices bound an edge if the curves they represent intersect *minimally*, i.e. once) to the classical Farey graph. For the punctured torus $T = S^1 \times S^1 - \{x_0\}$ the complex of curves is the same, but the collection of elements of $\pi_1 T$ representing these curves is significantly larger. Here $\pi_1 T$ is isomorphic to a rank-two free group on a pair of generators represented by simple closed curves intersecting once. A result of Birman–Series gives a useful necessary condition for a word in these generators to represent a simple closed curve on T [1, Theorem 5.1].

In the non-orientable context, Scharlemann described the complexes of curves of low complexity surfaces [22]. The main result of the current paper, Theorem 3.3, follows up with a result for the punctured Klein bottle K analogous to Theorem 5.1 of [1]. Again $\pi_1 K$ is isomorphic to a free group on generators a and b represented by simple closed cuves intersecting once (see Figure 1). Theorem 3.3 characterizes the words in a and b represented by simple closed curves on K.

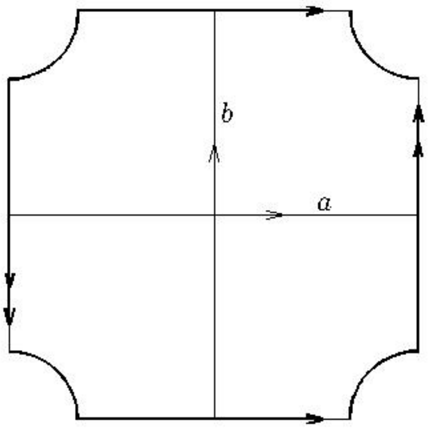

Fig. 1: A depiction of K where $\pi_1(K) = \langle a, b \rangle$

Our proof of this result employs the other key object mentioned above: the *mapping class group*, which, for a surface F, is the group $\mathcal{M}(F)$ of "homeomorphisms of F up to isotopy". Put more precisely, $\mathcal{M}(F)$ is the quotient of the group of homeomorphisms of F, where this group is equipped with the compact-open topology, by its component containing the identity map. It follows that two homeomorphisms of F represent the same mapping class in $\mathcal{M}(F)$ if and only if they are isotopic. (For an orientable surface F, what we have defined here is often called the *extended* mapping class group, with the unmodified term reserved for the classes of orientation-preserving homeomorphisms.)

The mapping class group is both a key tool in analyzing the topology of surfaces and an important object of study in its own right: for example, W.P. Thurston gave a classification of its elements, essentially rediscovering earlier work of Nielsen (see eg. [6, Chapter 13]), and a presentation (for F orientable) in joint work with Hatcher [10]. Harer's work [8] was devoted to determining its stable homology groups.

The (non-extended) mapping class group of an orientable surface has long been known to be generated by "Dehn twists" about its simple closed curves, see [14]. Korkmaz showed that if F is a non-orientable surface then a generating set for $\mathcal{M}(F)$ must also include "crosscap" and "boundary slides", and described a finite generating set for each such mapping class group [12]. In Section 2 we will describe these generators and slightly simplify Korkmaz' given generating set for $\mathcal{M}(K)$, where K is the punctured Klein bottle.

We also record the action of each generator of $\mathcal{M}(K)$ on $\pi_1 K$. In the parlance of Chapter 1 of Hatcher's *Algebraic Topology* [11], this amounts to fixing a basepoint $x \in K$ and, for a fixed representative ϕ of a generator for $\mathcal{M}(F)$, choosing an arc h in K joining x to $\phi(x)$. An automorphism of $\pi_1(K, x)$ is then given by $\beta_h \circ \phi_*$ (cf. [11, Prop. 1.5]). This automorphism depends on the choice of h and the choice of representative ϕ for its mapping class, but only up to conjugation in $\pi_1(K, x)$ (compare [11, Lemma 1.19]), so it is important to note that the "action" of the generators for $\mathcal{M}(K)$ recorded in Section 2 is well-defined only up to conjugacy.

An analogous discussion implies the well known fact that for any surface F there is a homomorphism

$$\mathcal{M}(F) \longrightarrow \mathrm{Out}(\pi_1 F),$$

the quotient of the automorphism group of $\pi_1 F$ by those induced by conjugation. This homomorphism records the action of a representative of each mapping class. It is an isomorphism if F is closed and orientable (cf. [6, Theorem 8.1]) but not necessarily so for punctured surfaces, even in the orientable case, as the collection of conjugacy classes representing punctures must be preserved. The non-orientable case introduces new obstructions, since, for example, no mapping class can take a one-sided curve (like a in Figure 1) to a two-sided curve (like b) or vice-versa. See Remark 1, where we compare the mapping class groups of the Klein bottle, punctured torus, and three-punctured sphere.

For an arbitrary surface F, $\mathcal{M}(F)$ acts on the set of isotopy classes of simple closed curves of F via

$$[\phi].[c] = [\phi(c)]$$

for a homeomorphism ϕ and simple closed curve c, where brackets denote isotopy classes. In Section 3 we first leverage the classification of surfaces, in Lemma 3.2, to produce a short list of representatives $\mathcal{C}_0$ for the orbits of the action of $\mathcal{M}(K)$ on the non-nullhomotopic simple closed curves of the punctured Klein bottle K. We then prove Theorem 3.3 by recursively describing the words in $\pi_1 K$ obtained from $\mathcal{C}_0$ by the $\mathcal{M}(K)$-action, using Section 2's description of the generators of $\mathcal{M}(K)$ and their action on $\pi_1 K$.

In Section 4 we apply Theorem 3.3 to describe a representation of $\pi_1 K$ to $\mathrm{PGL}_2(\mathbb{R})$ which is not injective, but which has no simple loop, i.e. an element represented by a simple closed curve, in its kernel. This provides a counterexample to the punctured Klein bottle case of the "simple loop conjecture for representations to $\mathrm{PGL}_2(\mathbb{R})$". The original simple loop conjecture, which was proved by Gabai [7], posits that every non-injective homomorphism between the fundamental groups of orientable surfaces has a simple loop in its kernel. Gabai's proof method suggests a natural extension of the conjecture to an analogous assertion for homomorphisms from a surface group to a three-manifold group. This conjecture is still open.

Yair Minsky asked in [20] whether the "simple loop conjecture for representations to $\mathrm{PSL}_2(\mathbb{C})$" holds; that is, whether every representation from an orientable surface group to $\mathrm{PSL}_2(\mathbb{C})$ has a simple loop in its kernel. If this were true then it would imply the simple loop conjecture for maps to fundamental groups of hyperbolic three-manifolds, since each of these represents faithfully into $\mathrm{PSL}_2(\mathbb{C})$. But it is false, as has now been shown independently by Cooper–Manning [3], Calegari [2], Louder [15], and Mann [16]. In particular, Mann showed that orientable surface groups have non-injective representations to $\mathrm{PSL}_2(\mathbb{R})$, and non-orientable surface groups to $\mathrm{PGL}_2(\mathbb{R})$, without simple loops in the kernel.

Mann left two low-complexity cases open in [16]: the punctured Klein bottle K and the non-orientable genus-three surface. (Here the non-orientable genus refers to the number of projective plane summands in a connected sum decomposition; in particular the Klein bottle is the non-orientable genus-two surface.) Theorem 4.1 addresses the case of K, providing a simple explicit criterion (and a family of examples that satisfy it) for showing that a solvable (and hence non-injective) representation $\pi_1 K \to \mathrm{PGL}_2(\mathbb{R})$ has no simple loop in its kernel.

2 Mapping Class Group of K

To give a description of the mapping class group of the punctured Klein bottle, K, we define the necessary diffeomorphisms to generate the mapping class group and provide notation for each. We then prove these particular maps are indeed the elements that generate the group.

The first is a diffeomorphism called the *Dehn twist*. It is supported in an oriented regular neighborhood of any simple closed curve. In the case of the punctured Klein bottle, the necessary Dehn twist is defined about the simple closed curve represented by $b \in \pi_1(K)$, which can be written explicitly as the map $S^1 \times I$ defined by $(\theta, t) \to (\theta + 2\pi t, t)$, since the curve represented by b is simple. Intuitively, this can be thought of by separating the surface along b, rotating one end by 2π and gluing the surface back together. We borrow notation from Korkmaz and denote the Dehn twist about b and its isotopy class t_b[12]. Below record the action of t_b on the generators, a and b, of the $\pi_1(K)$ group from Figure 1.

$$t_b : \begin{cases} a \mapsto ab \\ b \mapsto b \end{cases}$$

The *cross cap slide*, or $Y-$*homeomorphism* is a map defined on non-orientable surface with genus at least two. Consider the punctured Klein bottle as obtained from an open Mobius band M with one hole (ie. a missing open disk) by identifying antipodal points on the boundary of the hole. The map is induced by sliding the hole around the core of M, then again identifying points of its boundary [12]. Alternatively, If we consider K to be $S^1 \times I$ with its ends identified in opposite orientation, with one puncture as above, we can describe The $Y-$ *homomorphism* as induced by the reflection of the cylinder about $S^2 \times \frac{1}{2}$[13]. We reserve y to represent the y-homeomorphism and the associated isotopy class.

$$y : \begin{cases} a \mapsto a^{-1} \\ b \mapsto b \end{cases}$$

The last of the mapping class group generators is the diffeomorphism supported in a neighborhood of an orientation reversing simple closed curve defined by pushing the puncture once along the curve. This action is called the *boundary slide*, and acts trivially along the boundary of any puncture[12].

The boundary slide along γ_1 operates on the generators of $\pi_1(K)$ as follows, depicted in Figure 2:

$$w_1 : \begin{cases} a \mapsto a \\ b \mapsto b^{-1} \end{cases}$$

Our description of the mapping class group generators is a corollary to the following result of Korkmaz[12, Theorem 4.9]:

Theorem 2.1. *Let S be a non orientable surface of genus 2 with 1 puncture. The Pure mapping class group $\mathcal{MP}_{2,1}$ of S is generated by $\{t_b, y, w_1, v_1\}$, where w_1 and v_1 are the boundary slides aroung the curves γ_1 and γ_2 of Figure 2 respectively.*

Here think of the punctured Klein bottle as the punctured 2-sphere after removing 2 interior open discs d_1 and d_2 then identifying antipodal points on each boundary. The 2-sphere is the compactification of the cartesian plane. Let the puncture be the removal of the point z . Let γ_1 be the simple closed curve based at z traveling through d_1 and γ_2 be another simple closed

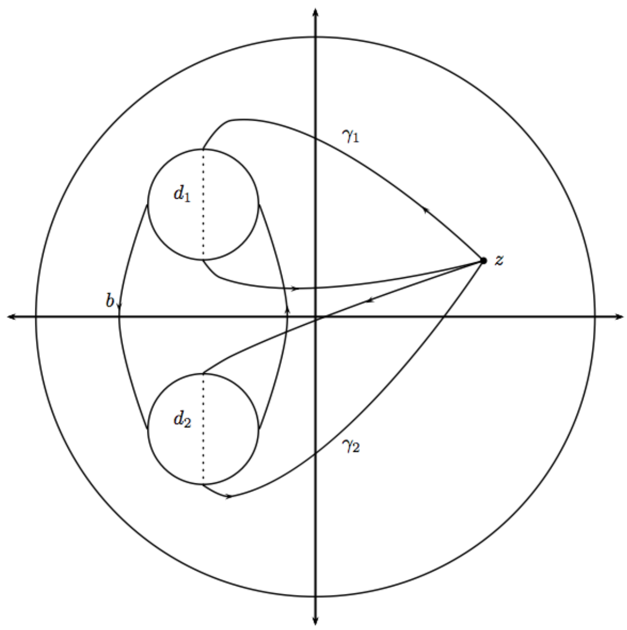

Fig. 2: Another model of K in the plane

curve based at z traveling through d_2. In this picture, a is represented by the boundary of d_2, and b is a path connecting d_1 and d_2, Figure 2. Let w_1 denote the boundary slide around γ_1 and let v_1 be the boundary slide around γ_2.

Lemma 2.2. *The mapping class group of K is generated by $\{t_b, y, w_1\}$.*

Proof. Since K has only one puncture, the mapping class group, and the pure mapping class group coincide. In Figure 3 we show $t_b(\gamma_1) = \gamma_2^{-1}$. It follows that the boundary slide around γ_1^{-1} is $t_b(\gamma_1)w_1t_b(\gamma_1)^{-1} = v_1^{-1}$. Thus there are three generators of the mapping class group $\{t_b, y, w_1\}$. □

Remark 1. Here we compare the mapping class groups of the punctured torus, the three-punctured sphere, and the punctured Klein bottle K, each a surface with fundamental group isomorphic to $\mathbb{F}^2$, the free group on two generators. For every surface F there is a commutative triangle of maps:

$$\begin{tikzcd} \mathcal{M}(F) \arrow[rr] \arrow[dr] & & \mathrm{Out}(\pi_1 F) \arrow[dl] \\ & \mathrm{GL}(H_1(F)) & \end{tikzcd}$$

The horizontal arrow here was described in the Introduction, and the other map from $\mathcal{M}(F)$ simply records the action of mapping classes on the first homology group $H_1(F)$. The map from $\mathrm{Out}(\pi_1 F)$ records its action on the abelianization of $\pi_1 F$, which we recall is isomorphic to $H_1(F)$, see eg. [11, Th. 2A.1]. (A standard exercise shows for an arbitrary group G that the action

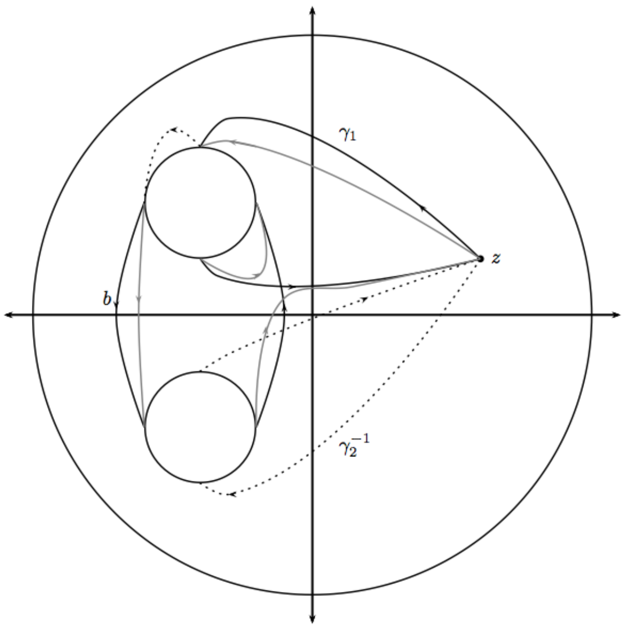

Fig. 3: $\gamma_2^{-1} = t_b(\gamma_1)$ after isotopy.

of its automorphism group $\mathrm{Aut}(G)$ on G induces an action of $\mathrm{Out}(G)$ on its abelianization $G/[G,G]$, where $[G,G]$ is the commutator subgroup.)

It is well known that all maps above are isomorphisms when $F = T$ is the punctured torus. (For the map $\mathcal{M}(T) \to \mathrm{GL}(H_1(T))$ see [6, §2.2.4]; for the map $\mathrm{Out}(\pi_1 F) \to \mathrm{GL}_2(\mathbb{Z})$ this goes back to Nielsen [21].) So in particular,

$$\mathcal{M}(T) \cong \mathrm{GL}_2(\mathbb{Z}) \cong \mathrm{Out}(\mathbb{F}_2),$$

since $H_1(T) \cong \mathbb{Z}^2$ is the abelianization of $\mathbb{F}_2$. On the other hand, $\mathcal{M}(\mathcal{F})$ is *finite* when F is the three-punctured sphere, see [6, Prop. 2.3].

The case $F = K$ of the punctured Klein bottle lies in between these extremes. On the one hand, $\mathcal{M}(K)$ is infinite, since t_b acts as the infinite-order matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ on $H_1(K)$. On the other, no element of $\mathcal{M}(K)$ acts as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, for example. This is because, as observed in the Introduction, a is represented by a *one-sided* simple closed curve — for which every small regular neighborhood is homeomorphic to a Möbius band — whereas b is *two-sided*, with annular regular neighborhoods. So no homeomorphism of K takes one to the other.

3 Classification of Simple Closed Curves on K

To classify simple closed curves on K, we rely heavily on the classification theorem for compact 2-manifolds, Stated in Massey, [17, Theorem 5.1]:

Theorem 3.1. *Any compact 2-manifold is homeomorphic to either a sphere, connected sum of tori, or a connected sum of projective planes* $\mathbb{R}\mathbf{P^2}$.

We say that a sphere has genus zero, and that a connected sum of tori or projective planes has genus equal to the number of summands. We declare the genus of a surface with boundary to be that of the closed surface obtained by joining a disk to each boundary component. It should also be noted that genus is a topological invariant.

This result allows to distinguish possible simple closed curves γ by Identifying the resultant surfaces after cutting along the specified curve.

Lemma 3.2. *Let $\mathcal{C}_0 = \{a, b, a^2, ab^{-1}a^{-1}b^{-1}\}$. Every simple closed curve, γ, on K is mapping class group equivalent to an element $c_0 \in \mathcal{C}_0$ or its inverse.*

It is useful to recall the Euler characteristic of an orientable surface is given by $\chi(S) = 2 - 2g - i$ or $\chi(S) = 2 - g - i$ in the non-orientable case, where g is the genus, and i is the number of boundary components. The Euler characteristic is invariant under the removal of any simple closed curve, $\chi(S)$=$\chi(S - \gamma)$.

Proof. We consider the following cases:

(i) γ separates K into two orientable surfaces S_1 and S_2.

This case is clearly impossible since a connect sum of two orientable surfaces must be orientable.

(ii) γ separates K into two non-orientable surfaces S_1 and S_2.

We let S_i have i boundary components which restricts the genera to the relation $g_1 + g_2 = 2$, where g_i is a natural number. As such, $g_1 = g_2 = 1$ is the only possibility. This implies S_1 is a Mobius band and S_2 is the Mobius band with one puncture. This is produced by cutting K along the closed curve represented by a^2 in the fundamental group.

(iii) γ separates K into an orientable surface S_1 and a non-orientable surface S_2.

(a) S_1 is punctured.

Again the relation $\chi_1 + \chi_2 = -1$ restricts the genera to satisfy $2g_1 + g_2 = 2$ with $g_1 \geq 0$ and $g_2 \geq 1$. When $g_1 = 0$, $g_2 = 2$, which implies S_1 is the surface of Euler characteristic zero with two boundary components, i.e. the Annulus. S_2 is the surface of genus 2 with Euler characteristic -1 and a single boundary component., i.e. the punctured Klein bottle. The disjoint union of the annulus and punctured Klein bottle arise by cutting K along $ab^{-1}a^{-1}b^{-1}$ in our representation of $\pi_1(K)$.

(b) S_2 is punctured.

When the puncture is on the non-orientable surface, the same relation on the genera holds. When $g_1 = 0$, S_1 is a disk, so as in case (i), is not considered.

(iv) γ is non-separating and the resulting surface is orientable.

(a) S has 2 boundary components.

The Euler characteristic is given by $\chi = 2-2g-2 = -1$. There is not natural number satisfying the equation, so this case is not possible.

(b) S has 3 boundary components.

γ cuts K such that the resulting surface has 3 boundary components, and satisfies $\chi = 2-2g-3 = -1$ which forces g to be zero, and the surface to be the 3-holed sphere, attained by cutting along the generator b of $\pi_1(K)$.

(v) γ is non-separating and the resulting surface is non-orientable.

(a) S has 2 boundary components.

γ cuts K such that the resulting surface has 2 boundary components, and satisfies $\chi = 2-g-2 = -1$, which forces g to be 1, so the surface to be the once punctured Mobius band and is produced by Cutting K along the curve represented by $a \in \pi_1(K)$.

(b) S has 3 boundary components.

The Euler characteristic is $\chi = 2-g-2 = -1$, which implies $g = 0$. The genus of a non-orientable surface must be greater than or equal to 1, so this case is disqualified.

□

Theorem 3.3. *Let $\mathcal{C}$ be the set of words of the following following form, and their inverses:*

(i) $a^{\pm 1}$, $a^{\pm 2}$, or $b^{\pm 1}$

(ii) $ab^{-1}a^{-1}b^{-1}$

(iii) ab^n or ab^nab^n for $n \in \mathbb{Z}$

Every simple closed curve on K represents a word in $\mathcal{C}$, and every word in $\mathcal{C}$ is represented by a simple closed curve.

This comports with the description of the punctured Klein bottle's complex of curves given on [22, pp. 175–176], with $\bar{a}$ there in the role of b here and vice-versa. (Note that the boundary is ignored in the complex of curves of [22], as are the squares of one-sided curves.)

We use the following lemmas to prove Theorem 3.3.

Lemma 3.4. *The mapping class group preserves the structure of $\mathcal{C}$, i.e. for any element $[\phi]$ in the mapping class group, and any word form $c \in \mathcal{C}$, $\phi(c)$ is conjugate into $\mathcal{C}$.*

Proof.
Case 1: $t_b(c) \in \mathcal{C}$ for any $c \in \mathcal{C}$.
By definition of the Dehn twist around b, $t_b(a) = ab$, $t_b(a^2) = abab$, and $tb(b) = b$.

On the boundary curves beginning with a, t_b acts trivially; $t_b(ab^{-1}a^{-1}b^{-1}) = ab^{-1}a^{-1}b^{-1}$.

Applying t_b to $a^{-1}b^n$ gives $b^{-1}a^{-1}b^n$, which is conjugate to $a^{-1}b^{n-1}$, and t_b applied to $a^{-1}b^na^{-1}b^n$ equals $b^{-1}a^{-1}b^{n-1}a^{-1}b^n$ which is conjugate to $a^{-1}b^{n-1}a^{-1}b^{n-1}$. Finally, t_b maps the elements ab^n and ab^nab^n to ab^{n+1} and $ab^{n+1}ab^{n+1}$ respectively.

Case 2: $w_1(c) \in \mathcal{C}$ for any $c \in \mathcal{C}$.
Our boundary slide fixes a and takes b to its inverse, so acting on any word in $\mathcal{C}$ will

preserve the structure of the product exchanging powers of b for their negative.

Case 3: $y(c) \in \mathcal{C}$ for any $c \in \mathcal{C}$. Similarly, the Y-homeomorphism fixes b and takes a

to its inverse, so the argument in the previous case applies, which concludes this Lemma. □

Lemma 3.5. *For any $c \in \mathcal{C}$ there is a mapping class group element $[\phi]$ such that $\phi(c)$ is conjugate to some $c_0 \in \mathcal{C}_0$ or its inverse.*

Proof. Clearly $y : a \mapsto a^{-1}$, $y : a^{-2} \mapsto a^2$, and $w_1 : b \mapsto b^{-1}$, and it is obvious the maps hold in the other direction as well.

The boundary slide takes $ab^{-1}a^{-1}b^{-1}$ to $aba^{-1}b$ and again can be applied in the opposite direction. An analogous statement is true for w_1 acting on $a^{-1}b^{-1}ab^{-1}$ and $a^{-1}bab$.

The Dehn twist around b maps $a \mapsto ab$ and by induction, $t_b^n(a) = ab^n$. and similarly, $t_b : a^2 \mapsto abab$ and again, $t_b^n(a^2) = ab^nab^n$ follows by induction. Applying y to ab^nab^n and ab^n yields $a^{-1}b^na^{-1}b^n$ and $a^{-1}b^n$ respectively. Acting on ab^nab^n, ab^n, $a^{-1}b^na^{-1}b^n$ and $a^{-1}b^n$ with the boundary slide produces a representation of the remaining classes of words in $\mathcal{C}$. □

Now, we prove Theorem 3.3.

Proof. Let c be a simple closed curve on K. Lemma 3.2 implies there is a mapping class group element such that $\phi(c)$ is conjugate to some $c_0 \in \mathcal{C}_0$. It follows by Lemma 3.4 that $\phi^{-1}(c_0)$, which represents c is conjugate to a word in $\mathcal{C}$. Since every word in $\mathcal{C}_0$ represents a simple closed curve, and mapping classes take simple closed curves to simple closed curves, Lemma 3.5 implies every word in $\mathcal{C}$ represents a simple closed curve. □

Thus, we have an explicit description of Simple closed curves on K up to inner automorphism, which corresponds to words permitted in $\mathcal{C}$.

4 Counterexample to the Simple Loop Conjecture for $PGL(2,\mathbb{R})$

In this section we rely on a representation of $\pi_1(K)$ in $PGL(2,\mathbb{R})$. Since our simple closed curves represented by a and b freely generate $\pi_1(K)$, any choice of α and β such that $\alpha, \beta \neq 0$ defines a representation $\rho : \pi_1(K) \to PGL(2,\mathbb{R})$.

$$\rho(a) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} \beta & 1 \\ 0 & \beta^{-1} \end{pmatrix}$$

This representation allows for the extension of [16, Theorem 1.3] to the punctured Klein bottle as follows.

Theorem 4.1. *Let K be the punctured Klein bottle; Let α and β satisfy $\alpha^k\beta^l \neq \pm 1$ for any integers $k, l \in \mathbb{Z}\backslash\{0\}$ (note that it is sufficient to choose α and β to be relatively prime integers). Then*

$$\rho : \pi_1(K) \to PGL(2,\mathbb{R})$$

satisfies

1. *ρ is not injective.*
2. *If $\rho(\alpha) = \pm I$, then α is not represented by a simple closed curve.*
3. *If α is represented by a simple closed curve, then $\rho(\alpha^k) \neq I$ for any $k \in \mathbb{Z}\backslash\{0\}$*

Proof. To show the representation is non-injective, we first show $\pi_1(K)$ is not solvable since it has a quotient which is not solvable. This fact follows since there exists a homomorphism $f : \pi_1(K) \to A_5$ where $A_5 = \langle(123),(345)\rangle$ the alternating group of even permutations on 5 elements defined by $a \mapsto (123)$ and $b \mapsto (345)$[5, Pg. 111, Exercise 16]. The group A_5 is non commutative and simple. If we assume for a contradiction that A_5 is solvable, by simplicity, the only composition series is $A_5 \trianglerighteq 1$, which implies the quotient group $A_5/1 = A_5$ is abelian since quotient groups of consecutive terms of a composition series are abelian. This is our contradiction, thus A_5 is not solvable. It is a common fact that a quotient of a solvable group is solvable, which by contrapositive, shows $\pi_1(K)$ is not solvable.

Our representation ρ of $\pi_1(K)$ in contrast is 2 step solvable since any element X in

$$G^1 = [PGL(2,\mathbb{R}), PGL(2,\mathbb{R})]$$

is of the form,

$$X = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad X^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$

. And by direct computation, we see that any element in $G^2 = [G^1, G^1]$ is given by a product

$$XYX^{-1}Y^{-1} = \begin{pmatrix} 1 & x+y-x-y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, ρ has a composition series terminating in the trivial group. This implies ρ is solvable and non-injective proving 1 from Theorem 4.1.

To show consequences 2 and 3 of Theorem 4.1, it is enough to note that products of upper triangular matrices are upper triangular, and as a consequence have $1,1$-entries in the form of words in $\mathcal{C}$ prior to reduction in $\mathbb{R}$. Since no word form in $\mathcal{C}$ reduces to a nontrivial product of α and β or their inverses in $\mathbb{R}$, the top left entry of any matrix representing a simple closed curve in $\mathrm{PGL}(2,\mathbb{R})$ is never of the form $\alpha^k\beta^l$ with k and l both 0. Thus, no matrix representation or any power of a simple closed curve can be the identity. No power of simple closed curve is in the kernel of our representation ρ. ☐

The counter example to the simple loop conjecture can be further extended to the case of the non-orientable genus 3 surface, and is addressed in detail in [4, Proposition 3.2].

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