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The configuration space of the three dimensional Lens space $L(7,2)$ and its model

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Abstract

We study the algebraic topology of the two point configuration space on the lens space $L(7,2)$, describing an explicit finite dimensional model for the algebra of De Rham differential forms on the universal cover of the two point configurations space $\Omega^*_{DR}(\tilde{F}_2(L(7,2)))$.

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1 Introduction

Rational (or real) homotopy theory provides a rare instance in topology where spaces, which one takes to be simply connected, can be completely understood by an algebraic model once we only consider the non-torsion invariants of the space. The topological dictionary and the algebraic dictionary are completely equivalent in this case. This theory is vast and we only refer to the main reference on the subject [7].

To better explain what we do in this paper, we must say more about these algebraic models which always come in the form of *commutative graded differential algebras*, or CGDAs for short. The differential is denoted by d. For a smooth differential manifold M, the deRham differential forms $\Omega^*_{DR}(M)$ form a CDGA over the real numbers and this determines the real homotopy type of M (in particular, the ranks of the homotopy groups). This CDGA is however generally very large and one seeks to find a smaller more manageable CDGA quasi-isomorphic to $\Omega^*_{DR}(M)$, thus yielding the same real homotopy type.

More precisely, a CDGA A is called a model for $\Omega^*_{DR}(M)$ if there is a third intermediate CDGA B mapping homomorphically to both

$$A \longleftarrow B \longrightarrow \Omega^*_{DR}(M)$$

with the arrows inducing an isomorphism in cohomology. All algebras in our case are defined over $\mathbb{R}$.

Our goal in this paper is to exhibit a finite dimensional model (A,d) for the De Rham differential forms on the universal cover of a particular three dimensional manifold which is the Lens space $L(7,2)$, obtained as a suitable quotient of the three sphere S^3 (definitions in the text). This Lens space; which is one in a family of spaces $L(p,q)$ with p,q relatively prime, was used by Longoni and Salvatore [13] to provide the only known counterexample to the conjecture that the diagonal complement of a closed oriented manifold, or equivalently its *two points configuration space*, is a homotopy invariant. In constructing the counterexample, one takes advantage of the fact that some Lens spaces are known to be homotopy equivalent but not homeomorphic. Note that the difficulty in dealing with this conjecture is that plain homological or homotopy group calculations cannot distinguish between the various diagonal complements associated to $L(p,q)$. There is a need to look for deeper invariants, like Massey products, and thus the need to go through universal covers to detect those subtle differences, as was done in the Longoni-Salvatore paper.

Denote by $\tilde{\mathcal{F}}_2(L(7,2))$ the universal cover of the Lens space $\mathcal{F}_2(L(7,2))$. Our main results are given in section 3 where the model is constructed and then in section 4 where we furthermore observe that the model is equivariant with respect to the action of the fundamental group. As an application of our construction, we recover the non-trivial massey product in the cohomology of $\mathcal{F}_2(L(7,2))$ discovered by Longoni and Salvatore.

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1.1 Configuration spaces

The configuration space of n points in a manifold M is the space

$$\mathcal{F}_n(M) := \{(x_1,\ldots,x_n) \in M^{\times n} : x_i \neq x_j \text{ for } i \neq j\}. \tag{1.1}$$

or equivalently the complement

$$\mathcal{F}_n(M) = M^n - \Delta,$$

where Δ denotes the diagonal

$$\Delta = \{(x_1, \dots, x_n) |\ x_i = x_j \text{ for some } i \neq j\}.$$

One thinks of $\mathcal{F}_n(M)$ as all n-tuples of points of M that are pairwise distinct.

Configuration spaces have been studied in many different situations. For example, $\mathcal{F}_n(\mathbb{R}^3)$ is the natural setting for the n-body problem (for a history of this topic see [4]). Configurations of points in $\mathbb{R}^3$ that are required to adapt to a given geometry are used in *robotics* and the *piano mover's problem* (for more details on this topic, see [5]). In mathematical physics, configuration spaces have been considered in relation to gauge theory, gravity or particle physics. In topology and geometry, these spaces enter as a fundamental tool in the study of *spaces of functions* and *moduli spaces*.

Exemple 1.1. The best-known and simplest non trivial examples are the configuration spaces $\mathcal{F}_n(\mathbb{R}^2)$, whose fundamental groups $P_n = \pi_1(\mathcal{F}_n(\mathbb{R}^2))$ are the so-called pure braid groups. The classical *Artin braid groups* appear as the fundamental groups of the unordered configuration spaces $B_n = \pi_1(\mathcal{UF}_n(\mathbb{R}^2))$, where $\mathcal{UF}_n(\mathbb{R}^2)$ is defined to be the quotient of $\mathcal{F}_n(\mathbb{R}^2)$ by the natural action of the symmetric group S_n which permutes the points in $\mathbb{R}^2$. The braid description is used to "visualize" paths in the configuration space, thus providing a very intuitive description.

1.2 Lens Spaces

The term "lens space" usually refers to a specific class of 3-manifolds, albeit these can be defined in higher dimensions. They were introduced for the first time by H. Tietze in 1908. There is more than one way to construct lens spaces in the 3-dimensional case. One way is to take the quotient of the unit 3-sphere S^3 by an action of a cyclic group. Another way is by gluing two solid tori together via a homeomorphism of their boundary.
In the rest of the paper we identify $\mathbb{Z}_p$ to the group of p^{th} complex roots of unity generated by

$$\zeta = e^{2\pi i/p}.$$

Let S^3 the unit 3-sphere viewed as a submanifold of $\mathbb{C}^2$;

$$S^3 \cong \{(z_1, z_2) \in \mathbb{C}^2 |\ |z_1|^2 + |z_2|^2 = 1\}.$$

Define for each $q \in \mathbb{Z}_p$ relatively prime to p define the action of $\mathbb{Z}_p$ on S^3 such that the generator acts by the map ζ_q with

$$\zeta_q(z_1, z_2) = (\zeta z_1, \zeta^q z_2)$$

where we denote by z_1 and z_2 the complex coordinates of S^3.

Definition 1.2. Let $(p, q) \in \mathbb{Z}$ be relatively prime. The lens space $L(p, q)$ is defined to be the quotient $S^3/\langle\zeta_q\rangle$.

Observation 1.3. The projection $S^3 \to L(p, q)$ is a covering space projection since the action of $\mathbb{Z}_p$ is free. Only the identity element has fixed points in S^3, and for each point $z_j \in S^3$, $e^{2\pi ikq/p} z_j \neq z_j$, for all $0 < k < p$. This is a consequence of the assumption that $mcd(p, q) = 1$.

Because S^3 is simply connected, it follows immediately that S^3 is the universal covering of $L(p, q)$ and that $\pi_1(L(p, q)) \cong \mathbb{Z}_p$. An important theorem of Reidemeister [18] classifies the Lens spaces up to homeomorphism and homotopy equivalence.

Theorem 1.4. The lens spaces $L(p, q)$ and $L(p, q')$ are homotopy equivalent if and only if for some $m \in \mathbb{Z}_p$ we have that:

$$q' \equiv \pm q m^2 \bmod p.$$

They are homeomorphic if and only if

$$q' \equiv \pm q^{\pm 1} \bmod p.$$

2 Two point configuration space of lens spaces

In this section we will discuss the cohomology of the universal cover of two point configuration spaces of lens spaces, as calculated in [13], and we present a list of generators of the relative homology groups $H_*(S^3 \times S^3, \Delta_q)$, where Δ_q is defined below, and describe their respective Lefshetz duals.

The universal cover of the two point configuration space $\mathcal{F}_2(L(p, q))$ is given as follows

$$\tilde{\mathcal{F}}_2(L(p, q)) = \{(x, y) \in S^3 \times S^3 : x \neq \zeta_q^k y \quad \forall\, k \in \mathbb{Z}_p\}.$$

For each $q = 0, \dots, p-1$, define

$$\Delta_q := \bigcup_{k \in \mathbb{Z}_p} \Delta_q^k$$

where Δ_q^k is the image of the following embedding of S^3

$$S^3 \longrightarrow S^3 \times S^3$$
$$\omega \longmapsto (\omega, \zeta_q^k \omega).$$

If we write $\omega = (x_1, x_2)$ in complex coordinates, then $(\omega, \zeta_q^k \omega) = ((x_1, x_2), (\zeta^k x_1, \zeta^{kq} x_2))$, and we can view $\tilde{\mathcal{F}}_2(L(p, q))$ as the subset of $\mathbb{C}^2 \times \mathbb{C}^2$ given as follows

$$\{((z_1, z_2), (z_3, z_4)) \mid (z_1, z_2) \neq (\zeta^k z_3, \zeta^{qk} z_4), k \in \mathbb{Z}_p\}.$$

2.1 Representatives in homology and duality

From now on we denote $I(k)$ the open interval $(k-1,k)$ and fix the positive orientation of $I(k)$ as the canonical one.We also choose the orientation of S^3 to correspond to the orientation at the point $(0,1)$ given by the basis tangent vectors $(i,0),(1,0)$ and $(0,i)$. In [13], that authors computed the following cohomology groups

$$H^*(\tilde{\mathcal{F}}_2(L(p,q))) = \begin{cases} \mathbb{R}^6 & * = 2 \\ \mathbb{R} & * = 3 \\ \mathbb{R}^6 & * = 5. \end{cases}$$

where the cup product of the two dimensional classes with the three dimensional class give the five dimensional classes. By Lefshetz duality we have the isomorphisms

$$H_{6-*}(S^3\times S^3, \bigcup_{k\in\mathbb{Z}_p} \Delta_q^k) = H^*(\tilde{\mathcal{F}}_2(L(p,q))),$$

for each $*=1,\dots,6$. The explicit generators of this intersection homology ring are listed and labeled as follows:

$*$	H_*
4	$[A_k]$
3	$[S]$
1	$[S\cap A_k]$

where the indices k have to be read modulo p. These generators are represented by submanifolds and their (transversal) intersections. We need make this description explicit, and to that end we recall a definition from M.S. Miller [15].

Definition 2.1. Given a map between topological spaces $f: X\times Y\to X$, the *track* of f, denoted $\Gamma(f)$, is the image of $\pi_1\times f:(X\times Y)^2\to X\times X$ where π_1 is the canonical projection on the first factor.

Definition 2.2. The submanifold A_k is the track of the following map

$$\alpha_k : S^3\times I(k) \longrightarrow S^3$$
$$(\omega,s)\longmapsto \zeta_q^s\ \omega,$$

With prior notation $A_k=\Gamma(\alpha_k)$.

Equivalently using complex coordinates $\omega=(z_1,z_2)$ we have:

$$\begin{aligned} A_k &= \{(z_1,z_2);(\zeta^s z_1,\zeta^{qs}z_2) : s\in I(k)\} \\ &\subset S^3\times S^3\subset \mathbb{C}^2\times\mathbb{C}^2. \end{aligned}$$

To prove that the classes $[A_k]$ generate $H_4(S^3\times S^3,\Delta_q)$, it is enough to consider a portion of the long exact sequence of the pair $(S^3\times S^3,\Delta_q)$. To define the homological relative classes we have to consider the closure of the manifold A_k denoted by $\bar{A}_k$. The long exact sequence restricts to

$$\begin{array}{l} H_4(S^3\times S^3)=0 \\ \downarrow \\ H_4(S^3\times S^3,\Delta_q)\xrightarrow{\partial} H_3(\Delta_q)\longrightarrow H_3(S^3\times S^3)\cdots \end{array}$$

$$\partial : [\bar{A}_k] \longmapsto [\Delta_q^k]-[\Delta_q^{k-1}].$$

the algebraic boundary ∂ is deduced from the geometric boundary of $\bar{A}_k$ which is exactly $\Delta_q^k\cup\Delta_q^{k-1}$. The exact sequence takes then the form

$$0\longrightarrow H_4(S^3\times S^3,\Delta_q)\xrightarrow{\partial}\mathbb{Z}^p\xrightarrow{\phi}\mathbb{Z}\oplus\mathbb{Z}$$

where the map ϕ is given by

$$(v_1,\dots,v_p)\mapsto(\sum_{i=1}^p v_i,\sum_{i=1}^p v_i).$$

The kernel of ϕ is isomorphic to $\mathbb{Z}^{p-1}$ and by the injectivity of ∂ we have that $H_4(S^3\times S^3,\Delta_q)\cong\mathbb{Z}^{p-1}$. The generators of $H_4(S^3\times S^3,\Delta_q)$ are exactly $[\bar{A}_k]$ with the relation $\sum_{k\in\mathbb{Z}_p}[\bar{A}_k]=0$. To represent the generator $[S]\in H_3(S^3\times S^3,\Delta_q)$ we define S to be the image of the embedding $S^3\longrightarrow S^3\times S^3$ constant on the first coordinate sending $\omega\mapsto((1,0),\omega)$. For the generators of $H_1(S^3\times S^3,\Delta_q)$ we must consider the intersection $S\cap\bar{A}_k$ given by

$$S\cap\bar{A}_k=\{(1,0);(\zeta^s,0):\ s\in \bar{I(k)}\}.$$

We notice that the intersection is always transversal, indeed taking $w=(w_1,w_2)\in\tilde{\mathcal{F}}_2(L(p,q))$ we have that the following vectors are linearly independent and generate the tangent space at w respectively of the manifolds A_k and S:

$$\begin{aligned} T_w(A_k) = span\{&((0,0),(i\zeta^s,0));((0,1),(0,\zeta^{qs})); \\ &((0,i),(0,i\zeta^{qs}));((i,0),(i\zeta^s,0))\}, \end{aligned}$$

$$\begin{aligned} T_w(S) = span\{&((0,0),(i\zeta^s,0));((0,0),(0,\zeta^{qs})); \\ &((0,0),(0,i\zeta^{qs}))\}. \end{aligned}$$

The space $T_w(A_k)+T_w(S)$ has dimension six and this proves the transversality of the intersection.

We summarize the calculation in the following table, where the generators in H_{6-*} are the Lefshetz dual to the generators in H^*:

$*$	H^*	H_{6-*}	$6-*$
2	$[a_k]$	$[A_k]$	4
3	$[\eta]$	$[S]$	3
5	$[\eta a_k]$	$[S\cap A_k]$	1

where the indices k have to be read modulo p.

2.2 Intersection and transversality of the manifolds A_k

In this section we will discuss the intersections of the manifolds A_k. The intersections of more than three manifolds A_k is empty. From now on we denote by $(a,b)_{S^1}$ the projection of the interval $(a,b) \subset \mathbb{R}$ onto the quotient $\mathbb{R}/p\mathbb{Z}$.

Definition 2.3. Let j be an integer modulo p. Define j to be a q-covering if

$$j \equiv q^{-1}m \pmod p \qquad \text{for } |m| \in (0,q)_{S^1}.$$

Definition 2.4. Let j be a q-covering. Then $k \in \mathbb{Z}_p$ is an interloper of j if

$$\begin{cases} k \in [j,p]_{S^1} & \text{for} \quad m > 0 \\ k \in [0,j]_{S^1} & \text{for} \quad m < 0. \end{cases}$$

If j is not a q-covering then there are no interlopers of j.

In [15] section III.2, it was proved that if j is not a q-covering then the submanifold $A_k \cap A_{k+j}$ is empty and that

Proposition 2.5. If j is a q-covering and $I_\tau = (qI(k))_{S^1} \cap (qI(k+j))_{S^1}$, then the intersection $A_k \cap A_{k+j}$ is non-empty, transversal and homeomorphic to $S^1 \times I_\tau$.

In order to calculate the triple intersections we consider the submanifold $X_{ij} \cap A_k$ where X_{ij} is the manifold such that $\partial X_{ij} = A_i \cap A_j$. We see that $A_k \cap \partial X_{ij} = A_k \cap A_i \cap A_j$.

Proposition 2.6. [15] Let j and j' be q-covering, so $j \equiv mq^{-1}$, and $j' \equiv m'q^{-1}$ with $|m|$, $|m'| \in (0,q)_{S^1}$. Then the triple intersection $A_1 \cap A_{1+j} \cap A_{1+j'}$ is non empty if and only if $|m - m'|$ is in $(0,q)_{S^1}$.

We conclude with this

Proposition 2.7. Let j be q-covering and let $l \in \mathbb{Z}_p$. If l is also a q-covering assume that $|m - ql| \notin (0,q)_{S^1}$. Then

$$A_{1+l} \cap X_{i,j+1} = \begin{cases} \emptyset & \text{if } l \text{ is not an interloper of } j \\ A_{1+l} \cap S & \text{if } l \text{ is an interloper of } j \end{cases}$$

For the proof see again [15]. By the action of $\mathbb{Z}_p = \langle \zeta \rangle$ it is sufficient to consider the intersection with A_1.

3 The models

In this section we will discuss the explicit construction of the model $A(p,q)$. From now on we will fix the parameters $p = 7$ and $q = 2$, therefore the associated lens space is $L(7,2)$. The model $A(7,2)$ is a commutative differential graded algebra (CDGA) that is quasi-isomorphic to the universal cover of the two point configuration space of the lens space $\Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2))$.

Definition 3.1. Let A, B be two CDGA. We will say that they are quasi-isomorphic if there exists a CDGA C and a zigzag of homomorphisms f and g

$$A \xleftarrow{\ f\ } C \xrightarrow{\ g\ } B$$

such that the induced maps in cohomology are isomorphisms. In this case and as discussed in the introduction, we say that A is a model for B, and viceversa.

Definition 3.2. Let A be the free CDGA on the following generators

6	0
5	$\eta a_k, x_{ij}a_k$
4	$a_i a_j$
3	η, x_{ij}
2	a_k
1	

truncated in degree six, with the indices i,j,k all in $\mathbb{Z}_7$. We define the differential of the generators in this way:

$$d(a_k) = 0,$$
$$d(\eta) = 0,$$
$$d(x_{ij}) = a_i a_j.$$

Now we set the following relations: the elements x_{ij} are defined to be non-zero for all the indices $i < j$ such that $A_i \cap A_j \neq \emptyset$. By the proposition 2.5 we have that the only non-empty intersections are

$$A_i \cap A_{i+3} \text{ such that } i \in \mathbb{Z}_7,$$

so the only indices i,j for which x_{ij} is defined are in the set

$$G = \{(1,4);(2,5);(3,6);(4,7);(5,1);(6,2);(7,3)\} \subset \mathbb{Z}_7 \times \mathbb{Z}_7.$$

The relations for the product are:

$$\sum_{k \in \mathbb{Z}_7} a_k = 0, \tag{3.1}$$
$$x_{ij}a_k = 0 \qquad \text{for } k \text{ not interloper,} \tag{3.2}$$
$$x_{ij}a_i = 0 \qquad \forall\ (i,j) \in G, \tag{3.3}$$
$$x_{ij}a_k = \eta a_k \ \text{ for } (i,j) \in G \text{ and } k = i+1 \text{ or } k = i+2, \tag{3.4}$$
$$x_{ij}a_j = -\eta(a_{i+1} + a_{i+2})\ \forall\ (i,j) \in G, \tag{3.5}$$

Now we are able to give the following

Definition 3.3. Let $A(7,2)$ be the algebra A modulo the relations above.

By standard calculations we will verify that the cohomology with real coefficients of $A(7,2)$ is the same as that of $\tilde{\mathcal{F}}_2(L(7,2))$. Indeed we have

$$H^*(A(7,2)) = \begin{cases} \mathbb{R}^6 & * = 2 \\ \mathbb{R} & * = 3 \\ \mathbb{R}^6 & * = 5. \end{cases}$$

To prove that the CDGA $A(7,2)$ is quasi-isomorphic to $\Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2))$ we will construct a CDGA $B(7,2)$ and a zigzag of quasi-isomorphisms λ and μ between them

$$A(7,2) \xleftarrow{\lambda} B(7,2) \xrightarrow{\mu} \Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2)).$$

Definition 3.4. Let B the free graded differential algebra generated in degree one by z, in degree two by a_k, in degree three by η, x_{ij}, and in degree four by t_{ijl} , for $i,j,k \in \mathbb{Z}_7$. Define the differential to be

$$d(z) = \sum_{k\in\mathbb{Z}_7} a_k, \tag{3.6}$$
$$d(x_{ij}) = a_i a_j, \tag{3.7}$$
$$d(t_{ijk}) = (x_{ij} - \eta)a_k, \tag{3.8}$$

and zero on other generators.

Now we define the algebra $B(7,2)$ to be the algebra B modulo the following relations:

$$x_{ij} = 0 \qquad \forall\ (i,j) \notin G, \tag{3.9}$$
$$a_i a_j = 0 \qquad (i,j) \notin G, \tag{3.10}$$
$$t_{ijk} \neq 0 \qquad (i,j) \in G \text{ and } k \in \{i+1, i+2\}, \tag{3.11}$$
$$x_{ij}a_k = 0 \qquad (i,j) \in G \text{ and } k \in \{i, i+1, i+2, j\}, \tag{3.12}$$
$$t_{ijk}a_l = 0 \qquad \forall\ l \neq k. \tag{3.13}$$

For the sake of clarity, we list the generators of $B(7,2)$ up to degree six

7	$\cdots$
6	za_kx_{ij}, $\eta z a_k$, ηx_{ij}, $t_{ijk}a_k$, $a_i^2 a_j$, a_k^3
5	ηa_k, $x_{ij}a_k$, za_k^2, za_ia_j, zt_{ijk}
4	a_k^2, a_ia_j, ηz, zx_{ij}, t_{ijk}
3	η, x_{ij}, za_k
2	a_k
1	z

for appropriate indices i, j, k.

Observation 3.5. The algebra $B(7,2)$ unlike $A(7,2)$ is not truncated. In fact it has generators in each degree, but only a finite number of them. In degree six the triple product $a_ia_ja_k$ has been omitted because for each $i \neq j \neq k$ at least one couple between (i,j), (i,k), (j,k) is not in G, and so the corresponding product is zero.

By standard computations (which are omitted), we find that the algebra $B(7,2)$ has the required cohomology, which is the same as that of $A(7,2)$, up to degree five. In order to kill the cohomology in degree six we add cycles $w_1, \dots, w_m$ in degree five, so that the cohomology in lower degrees does not change and simultaneously we annihilate the cohomology in degree six. Moreover these generators do not produce cohomology in degree six because for different values of i the images of zw_i under the differential

$$\begin{aligned} d(zw_i) &= d(z)w_i + (-1)^{|z|} z\, d(w_i) \\ &= d(z)w_i + \dots = (\sum_{k\in\mathbb{Z}_7} a_k)w_i + \dots \end{aligned}$$

are linearly independent. This procedure is formal because the cohomology is finitely generated in every degree. Since $H^1(B(7,2)) = 0$ we can iterate this process for each degree introducing generators in degree l to annihilate the cohomology in degree $l+1$ for each $l \geq 5$. Therefore the three algebras $A(7,2), B(7,2), \Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2))$ have the same cohomology.

Next we analyze the algebra of differential forms $\Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2))$. As discussed in proposition 2.7 the intersection $A_k \cap X_{ij} = A_k \cap S$ for $k \in \{i, i+1, i+2, j\}$ and $(i,j) \in G$, so for these indices there exists a Thom form τ_{ijk} in degree four with support small enough such that $Supp(\tau_{ijk}) \supseteq X_{ij} \cap A_k$ and at the same time $Supp(\tau_{ijk}) \cap A_l = \emptyset$ for each $k \neq l$. Its differential is such that:

$$\begin{aligned} d : \Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2)) &\longrightarrow \Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2)) \\ \tau_{ijk} \quad &\longmapsto \quad (\chi_{ij} - \sigma)\omega_k, \end{aligned}$$

where the forms ω_k, χ_{ij} and σ are respectively Thom forms whose support is in a tubular neighborhood of the submanifolds A_k, X_{ij} and S.

To prove that the algebras $A(7,2)$ and $\Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2))$ as claimed, we have to map B to both of them via maps inducing isomorphism in cohomology. We start by defining the homomorphism

$$\lambda : B(7,2) \longrightarrow A(7,2),$$

$$\begin{aligned} \lambda(z) &= 0, \\ \lambda(a_k) &= a_k, \\ \lambda(x_{ij}) &= x_{ij}, \\ \lambda(\eta) &= \eta, \\ \lambda(t_{ijk}) &= 0. \end{aligned}$$

The induced map in cohomology is an isomorphism. On the other hand, we define the map

$$\mu : B(7,2) \to \Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2)),$$

as follows: since $\sum_{k\in\mathbb{Z}_7} \omega_k$ represents a trivial cocycle, there exists a differential form $\xi \in \Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2))$

such that $\sum_{k\in\mathbb{Z}_7}\omega_k = d\xi$. Therefore we set:

$$\begin{aligned}\mu(z) &= \xi,\\ \mu(x_{ij}) &= \chi_{ij},\\ \mu(\eta) &= \sigma,\\ \mu(t_{ijk}) &= \tau_{ijk}.\end{aligned}$$

By definition of μ the induced map in cohomology is an isomorphism as well. This implies the existence of a zigzag of quasi-isomorphism between $A(7,2)$ and $\Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2))$.

We observe that in $A(7,2)$ there exists a non-trivial Massey product, and so using the quasi- isomorphism μ we have the same non-trivial product in $\Omega^*_{DR}(\tilde{\mathcal{F}}_2(7,2))$ found in Longoni and Salvatore [13].

Proposition 3.6. The Massey product

$$\langle a_4, a_1, a_2 + a_6\rangle$$

is non-trivial.

Proof. We have that

$$d(x_{14}) = a_1 a_4,$$

and also

$$a_1(a_2 + a_6) = 0,$$

so by definition of the Massey product:

$$\begin{aligned}\langle a_4, a_1, a_2 + a_6\rangle &= x_{14}(a_2 + a_6)\\ &= x_{14}\ a_2 + x_{14}\ a_6\\ &= x_{14}\ a_2 = \eta\ a_2,\end{aligned}$$

that is non-trivial modulo $\langle \eta\ a_4, \eta\ (a_2 + a_6)\rangle$. □

4 Equivariance

In this final section we suitably modify the maps λ and μ so that they become equivariant with respect to the action of the second factor of the fundamental group of $\pi_1(\mathcal{F}_2(L(7,2))) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$.

Definition 4.1. (Equivariant map) Let G a group, X_1 and X_2 two G-sets and $\varphi : X_1 \to X_2$ a map between sets. Then φ is G-equivariant if it commutes with the G-action:

$$\varphi(g\cdot x) = g\cdot\varphi(x),$$

for all $x \in X_1$ and $g \in G$.

4.1 Equivariance of μ

As we stated in section 1.2 consider $\mathbb{Z}_7$ as the cyclic group of 7^{th} complex roots of unity generated by $\zeta = e^{2\pi i/7}$. It is enough to define the $\mathbb{Z}_7$-action on the elements z, a_k, x_{ij}, η, t_{ijk}. The group $\mathbb{Z}_7$ acts trivially on z and η, and by translation of indices on the other generators:

$$\begin{aligned}\zeta\cdot a_k &:= a_{k+1},\\ \zeta\cdot x_{ij} &:= x_{i+1,j+1},\\ \zeta\cdot t_{ijk} &:= t_{i+1,j+1,k+1}.\end{aligned}$$

We still need to define the action on the other generators in degree higher than five. To this aim we proceed as follows: since a $\mathbb{Z}_7$-action on a vector space is equivalent to a real $\mathbb{Z}_7$-representation, we consider the short exact sequence for $(i > 5)$

$$0 \to B^i(\tilde{B}(7,2)) \longrightarrow Z^i(\tilde{B}(7,2)) \xrightarrow{\pi} H^i(\tilde{B}(7,2)) \to 0,$$

where $\tilde{B}(7,2)$ is the subalgebra of $B(7,2)$ generated by the generators up to degree $i-2$, and we indicate respectively with B^i and Z^i the i^{th} co-boundaries and the i^{th} co-cycles. The vector space $H^i(\tilde{B}(7,2))$ is isomorphic to the direct sum of one-dimensional vector spaces W_j^{i-1} (equal to the number of added elements in degree $i-1$):

$$H^i(\tilde{B}(7,2)) \cong \oplus_j W_j^{i-1}.$$

For each i, the space W_j^{i-1} is the real vector space generated by the elements added in degree $i-1$ to kill the cohomology in degree i. The irreducible real $\mathbb{Z}_7$-representations are only four: the trivial one which is one-dimensional and three others of dimension two (up to isomorphism) induced by multiplication by the third roots of unity. Using Schur's lemma we deduce that each $\mathbb{Z}_7$-representation on $H^i(\tilde{B}(7,2))$ can be decomposed as a direct sum of irreducible real sub-representations of dimension one and two. So the problem to define the action is reduced to finding an equivariant section

$$s : H^i(\tilde{B}(7,2)) \to Z^i(\tilde{B}(7,2))$$

which we know exists by [10]. The existence of this section allows us to chose equivariant representatives in $Z^i(\tilde{B}(7,2))$. The $\mathbb{Z}_7$-action on $\Omega^*_{DR}(\tilde{\mathcal{F}}_2 L(7,2))$ is induced by the action of the fundamental group on the universal cover. We must appropriately modify the map μ in order to make it equivariant and so that the following diagram commutes:

$$\begin{array}{ccc} B(7,2) & \xrightarrow{\mu} & \Omega^*_{DR}(\tilde{\mathcal{F}}_2 L(7,2))\\ \zeta\cdot\downarrow & & \zeta\cdot\downarrow\\ B(7,2) & \xrightarrow{\mu} & \Omega^*_{DR}(\tilde{\mathcal{F}}_2 L(7,2)).\end{array}$$

Next we study the action $\zeta\cdot$ on the submanifold A_k of the second factor of the fundamental group. For each $k \in Z_7$ we have that:

$$\begin{aligned}\zeta\cdot A_k &= \{(z_1,z_2);(\zeta^{s+1}z_1,\zeta^{2s+2}z_2):\ s\in I(k)\}\\ &= \{(z_1,z_2);(\zeta^{t}z_1,\zeta^{2t}z_2):\ t\in I(k+1)\}\\ &= A_{k+1}.\end{aligned}$$

We define $\mu(a_k) = \omega_k$ where ω_k is the Thom form defined on a tubular neighborhood of A_k as in section 3. So since $\zeta \cdot A_k = A_{k+1}$ we have that $\zeta \cdot \omega_k = \omega_{k+1}$ by definition and so we have that the action in degree 2 is equivariant because

$$\zeta \cdot \mu(a_k) = \zeta \cdot \omega_k = \omega_{k+1} = \mu(a_{k+1}) = \mu(\zeta \cdot a_k),$$

for all $k \in \mathbb{Z}_7$. Since $\sum_{k\in\mathbb{Z}_7} \omega_k$ is a trivial cocycle in cohomology, there exists an element of degree one ξ such that $d\xi = \sum_{k\in\mathbb{Z}_7} \omega_k$. An arbitrary choice of such ξ does not assure the equivariance of this map because the action defined on $B(7,2)$ fixes the element z but this is not true for any choice of ξ. Concerning the one degree element z we redefine μ in this way:

$$\mu(z) := \frac{1}{7} \sum_{k\in\mathbb{Z}_7} \zeta^k \cdot \xi.$$

We have that $\zeta \cdot \mu(z) = \frac{1}{7} \sum_{k\in\mathbb{Z}_7} \zeta^{k+1} \cdot \xi = \mu(z) = \mu(\zeta \cdot z)$ for $k \in \mathbb{Z}_7$, that gives us the equivariance in degree one. Regarding degree three we proceed as follows: since the action on $B(7,2)$ fixes η we redefine μ. Let us consider the Thom form σ defined on a particular tubular neighborhood of a submanifold S, as in section 3, we define

$$\mu(\eta) := \frac{1}{7} \sum_{k\in\mathbb{Z}_7} \zeta^k \cdot \sigma,$$

so due to the cyclicity we have that

$$\zeta \cdot \mu(\eta) = \frac{1}{7} \sum_{k\in\mathbb{Z}_7} \zeta^{k+1} \cdot \sigma = \mu(\eta) = \mu(\zeta \cdot \eta),$$

and thus the related diagram is commutative. Also for the other elements in degree three x_{ij} with $(i,j) \in G$ we modify the map. By definition of $\zeta\cdot$ we have $\zeta \cdot X_{ij} = X_{i+1,j+1}$ so we define $\mu(x_{ij}) = \chi_{ij}$, where χ_{ij} is the Thom form defined on a particular tubular neighborhood of submanifold X_{ij}, as in section 3. Therefore ζ acts on the forms χ_{ij} by translation of (i,j): $\zeta \cdot \chi_{ij} = \chi_{i+1,j+1}$, and thus $\zeta \cdot \mu(x_{ij}) = \zeta \cdot \chi_{ij} = \chi_{i+1,j+1} = \mu(x_{i+1,j+1}) = \mu(\zeta \cdot x_{ij})$ that gives us the equivariance also in degree three. Finally in degree four we define the map μ on the generators by $\mu(t_{ijk}) = \tau_{ijk}$ for $(i,j) \in G$ and $k \in \{i+1, i+2\}$ where τ_{ijk} is the Thom form defined on a particular tubular neighborhood of the submanifold $X_{ij} \cap A_k$, as in section 3. We consider the action of ζ on this intersection and we have $\zeta \cdot \tau_{ijk} = \tau_{i+1,j+1,k+1}$. Thus we have the commutativity also in degree four, because

$$\begin{aligned}\zeta \cdot \mu(t_{ijk}) = \zeta \cdot \tau_{ijk} = \tau_{i+1,j+1,k+1} &= \mu(\tau_{i+1,j+1,k+1}) \\ &= \mu(\zeta \cdot \tau_{ijk}).\end{aligned}$$

4.2 Commutativity with the differential

We have also to check that the differential is equivariant. In the following we will check this only for the action defined on the algebra $B(7,2)$. Let us proceed to check this in degree one:

$$d(\zeta \cdot z) = d(z) = \sum_{k\in\mathbb{Z}_7} a_k = \zeta \cdot \Big(\sum_{k\in\mathbb{Z}_7} a_k \Big) = \zeta \cdot d(z).$$

For the elements a_k in degree two and for η in degree three we have that the equivariance holds because the differential on them is trivial. For the elements x_{ij} we proceed as follows:

$$d(\zeta \cdot x_{ij}) = d(x_{i+1,j+1}) = a_{i+1}a_{j+1} = \zeta \cdot (a_i a_j) = \zeta \cdot d(x_{ij}).$$

Finally we check the equivariance also for the elements in degree four:

$$\begin{aligned}\zeta \cdot d(t_{ijk}) &= \zeta \cdot ((x_i j - \eta)a_k) \\ &= (x_{i+1,j+1} - \eta)a_{k+1} \\ &= d(t_{i+1,j+1,k+1}) = d(\zeta \cdot t_{ijk}),\end{aligned}$$

because η is fixed by the action.

4.3 Equivariance of λ

We have also to define an action on the algebra $A(7,2)$ that we still call $\zeta\cdot$, such that λ is equivariant. Equivalently the following diagram:

$$\begin{array}{ccc} B(7,2) & \xrightarrow{\lambda} & A(7,2) \\ \zeta\cdot\downarrow & & \zeta\cdot\downarrow \\ B(7,2) & \xrightarrow{\lambda} & A(7,2) \end{array}$$

commutes.

Also in this subsection we will use the same action that we define on $B(7,2)$ and we will define another one on $A(7,2)$ as follow. For the element z in degree one we have:

$$\lambda(\zeta \cdot z) = \lambda(z) = 0,$$

and at the same time we have $\zeta \cdot \lambda(z) = 0$. For all $k \in \mathbb{Z}_7$, we define the action $\zeta \cdot a_k := a_{k+1}$ for the all elements a_k that in degree 2 of $A(7,2)$, so we have:

$$\lambda(\zeta \cdot a_k) = \lambda(a_{k+1}) = a_{k+1} = \zeta \cdot a_k = \zeta \cdot (\lambda(a_k))$$

(care must be taken as we previously defined the element a_k both in $A(7,2)$ and $B(7,2)$). In degree 3 we define the action on the element x_{ij} in this way $\zeta \cdot x_{ij} := x_{i+1,j+1}$, and with this definition we have that:

$$\lambda(\zeta \cdot x_{ij}) = \lambda(x_{i+1,j+1}) = x_{i+1,j+1} = \zeta \cdot x_{ij} = \zeta \cdot \lambda(x_i j)$$

Still in degree three, for the element η, we have that:

$$\lambda(\zeta \cdot \eta) = \lambda(\eta) = \eta = \zeta \cdot \eta = \zeta \cdot \lambda(\eta),$$

where, as usual we use the same name η for elements defined in the two different algebras. Finally in degree four for the elements t_{ijk} are sent to 0, and the equivariance is respected. The final statement that we have to check is the equivariance of the differential of $A(7,2)$. By similarity to what has been done in subsection 4.2 the calculation is omitted.

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