Graduate J. Math. **1** (2016), 28 – 42

Configuration Spaces of Graphs

Takeshi Unohara

Abstract

Given a graph Γ with a single essential vertex, we determine the fundamental group $\pi_1(C_n(\Gamma))$ of the configuration space $C_n(\Gamma)$ of Γ. This result is a generalization of Theorem 4.6 in [5]. We also discuss connectivity of configuration spaces of graphs. We use the CW complex model introduced in [15, 16] to study the configuration spaces of graphs.

MSC 2010. Primary 55R80; Secondary 20F36.

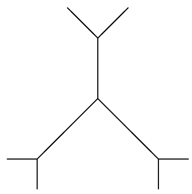
1 Introduction

The configuration space $C_n(X)$ of n distinct points in a topological space X is defined by

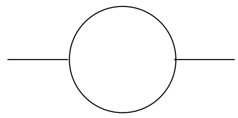
$$C_n(X) = \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Define an action of the symmetric group Σ_n on $C_n(X)$ by $x\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for $x \in C_n(X)$ and $\sigma \in \Sigma_n$. When X is a manifold (without boundary), the projection onto the first coordinate $p_1 : C_n(X) \to X$ is a fibration whose fiber over x is $C_{n-1}(X - \{x\})$ according to [6].

R. Ghrist first considered configuration spaces of graphs to investigate the problem of controlling robots in a factory. In general, graphs are not manifolds. When a topological space X is not a manifold, it is difficult to study the configuration space of X since the projection p_1 is not generally a fibration. In [8], Ghrist proved that configuration spaces of graphs are $K(\pi, 1)$-spaces (i.e. Eilenberg-MacLane spaces) and conjectured that the fundamental group $\pi_1(C_n(\Gamma))$ of the configuration space $C_n(\Gamma)$ of a graph Γ is a right-angled Artin group. Many people tried to prove this conjecture. Abrams and Ghrist in [2] showed that for the complete graph K_5 with 5 vertices and the complete bipartite graph $K_{3,3}$, the fundamental groups $\pi_1(C_2(K_5))$ and $\pi_1(C_2(K_{3,3}))$ are not right-angled Artin groups. Crisp and Wiest in [4] showed that for a graph Γ, the fundamental group $\pi_1(C_n(\Gamma)/\Sigma_n)$ embeds in a right-angled Artin group. Farley and Sabalka in [7] showed that for a tree Γ and $n = 2$ or 3, $\pi_1(C_n(\Gamma)/\Sigma_n)$ is a right-angled Artin group. Recently, Kim, Ko and Park in [10] showed that if a graph Γ contains neither T_0 nor S_0 and $n \geq 5$, then $\pi_1(C_n(\Gamma)/\Sigma_n)$ is a right-angled Artin group, where T_0 and S_0 are the following graphs



T_0



S_0

In general, configuration spaces of graphs are not CW complexes. But Abrams constructed a CW complex model.

Theorem 1.1 ([1]). *For a finite graph Γ with at least n vertices , define a CW complex $D_n(\Gamma)$ by*

$$D_n(\Gamma) = \bigcup_{\overline{e_{\lambda_i}} \cap \overline{e_{\lambda_j}} = \phi} (e_{\lambda_1} \times e_{\lambda_2} \times \ldots \times e_{\lambda_n}).$$

where $e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_n}$ are cells in Γ. Then $C_n(\Gamma)$ deformation retracts to $D_n(\Gamma)$ if and only if

1. *Each path between distinct vertices of valence not equal to two passes through at least $n - 1$ edges, and*
2. *Each path from a vertex to itself that cannot be shrunk to a point in Γ passes through at least $n+1$ edges.*

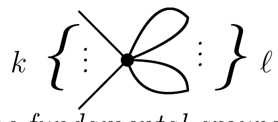
The configuration space $C_n(X)$ is not a CW complex but it is the "next best thing"; it is a *totally normal cellular stratified space* in the terminology of [15] and as discussed in the next sections. Associated to such a structure, there is a face category of which classifying space gives via the geometric realization functor a regular cell complex denoted $\mathrm{Sd}\,(C_n(X))$ (see section 4). What turns out to be quite useful is that $\mathrm{Sd}\,(C_n(X))$ is a strong deformation retract of $C_n(X)$ with respect to the action of the symmetric group and is quite a manageable complex to work with, in contrast with the Abrams' model where one needs to take finer and finer subdivisions and hence add more and more "cells".

In [5] and [14], Mukouyama showed the following theorem using Theorem 4.11.

Definition 1.2. Let Γ be a graph and $\varphi : D \to \overline{e}$ be an edge in Γ,

- An edge e is called *loop* if $D = [-1, 1]$ and $\varphi(-1) = \varphi(1)$.
- An edge e is called *bridge* if $D = [-1, 1]$, $\varphi(-1) \neq \varphi(1)$, and both $\varphi(-1)$ and $\varphi(1)$ are contained in more than one edge.
- An edge e is called *branch* if it is not a loop nor a bridge.
- For a vertex v, let b_v is the number of branches and bridges attached to v and let l_v is the number of loops attached to v.
- The number $b_v + 2l_v$ is called the *valency* at v.
- A vertex in Γ with valency 1 is called a *leaf*.

Theorem 1.3 ([5], [14]). *Let $\Gamma_{k,\ell}$ be the graph having k branches and ℓ loops attached to a single central vertex:*



Then the fundamental groups of the ordered and unordered configuration spaces of two points in $\Gamma_{k,\ell}$ are given by

$$\pi_1(C_2(\Gamma_{k,\ell})) \cong F_{m+1}$$
$$\pi_1(C_2(\Gamma_{k,\ell})/\Sigma_2) \cong F_{\frac{m}{2}+1}$$

where

$$m = (k+\ell)(k+3\ell-3),$$

and F_n denotes a free group of rank n.

We can push the technique further and generalize Theorem 1.3 to the configuration space with an arbitrary number of points not just two. Our main statement proven in section 8 takes the form:

Theorem 1.4. *For $(k,\ell) \neq (1,0),(2,0),(0,1)$ and a natural number n, the ordered and unordered configuration spaces of n distinct points in $\Gamma_{k,\ell}$ are homotopy equivalent to wedges of circles. Furthermore, we have*

$$\begin{aligned}\pi_1(C_n(\Gamma_{k,\ell})) &\cong F_{m+1}\\ \pi_1(C_n(\Gamma_{k,\ell})/\Sigma_n) &\cong F_{\frac{m}{n!}+1}\end{aligned}$$

where

$$m = \frac{(\ell+k+n-2)!}{(\ell+k-1)!}((2n-1)(\ell-1)+(n-1)k).$$

The proof of Theorem 1.4 proceeds in two steps. We first observe that the dimension of the Tamaki cellular model for the configuration space has dimension one (i.e. is a finite connected 1-dimensional cell complex). Therefore its fundamental group must be a free group with rank determined by the Euler characteristic χ. Then we compute χ of our model inductively.

We will also verify the following result in section 7.

Theorem 1.5. *Let Γ be a finite connected graph having a vertex v_0 of valency ≥ 3. Then $C_n(\Gamma)$ is path connected.*

Acknowledgments: I would like to thank Dai Tamaki for his advice and support. I would also like to thank Katsuhiko Kuribayashi for his comments and encouragement.

2 Classifying spaces of small categories

In this section, we recall facts about classifying spaces of small categories.

Definition 2.1. A category C is called a *small category* if both the class of objects in C, denoted C_0, and the class of morphisms in C, denoted C_1, are sets. For a morphism $f : x \to y$, the object x is called a *domain* and the object y is called a *range*. The set of morphisms from an object x to an object y is denoted by $C(x,y)$.

We recall the definitions of simplicial sets and their geometric realizations to construct classifying spaces.

A *simplicial set* X consists of a sequence of sets $X_0, X_1, \ldots$, a family of maps $\{d_i : X_n \to X_{n-1}\}_{0\leq i\leq n}$ and a family of maps $\{s_i : X_n \to X_{n+1}\}_{0\leq i\leq n}$ for each n satisfying some standard "simplicial" identities.

$$\begin{aligned}
&d_i \circ d_j = d_{j-1} \circ d_i && (i<j)\\
&d_i \circ s_j = s_{j-1} \circ d_i && (i<j)\\
&d_j \circ s_j = id = d_{j+1} \circ s_j &&\\
&d_i \circ s_j = s_j \circ d_{i-1} && (i>j+1)\\
&s_i \circ s_j = s_{j+1} \circ s_i && (i\leq j).
\end{aligned}$$

Let Δ^n be the standard n-simplex

$$\Delta^n = \left\{ (t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\}.$$

with maps $d^i : \Delta^{n-1} \to \Delta^n$ defined by

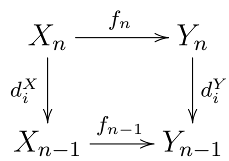
$$d^i(t_0, t_1, \ldots, t_{n-1}) = (t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n).$$

and maps $s^i : \Delta^n \to \Delta^{n-1}$ defined by

$$s^i(t_0, t_1, \ldots, t_n) = (t_0, t_1, \cdots, t_i + t_{i+1}, \ldots, t_n).$$

Example 2.2. Let X be a topological space. The set $\mathrm{Map}(\Delta^n, X)$ of continuous maps from the standard n-simplex $\Delta^n$to X is denoted by $S_n(X)$. Define maps $d_i : S_n(X) \to S_{n-1}(X)$ by $d_i(f) = f \circ d^i$. Define maps $s_i : S_{n-1}(X) \to S_n(X)$ by $s_i(f) = f \circ s^i$. Then $S(X) = (S_n(X), d, s)$ is a simplicial set. This simplicial set $S(X)$ is called the singular simplicial set of X.

Definition 2.3. Let X and Y be simplicial sets. A collection of maps $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}\cup\{0\}}$ is called a *simplicial map* if for any nonnegative number n and $i \in \{0, 1, \ldots, n\}$, the following diagrams are commutative



$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle s_i^X}\downarrow & & \downarrow{\scriptstyle s_i^Y} \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1}. \end{array}$$

The collection $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}\cup\{0\}}$ is denoted by $f : X \to Y$.

Definition 2.4. Let X be a simplicial set. The *geometric realization* of X is defined by

$$|X| = \left(\coprod_{n=0}^{\infty} X_n \times \Delta^n\right)\Big/\sim$$

where the relation $\sim$ is the equivalence relation generated by

$$\begin{aligned} (x, d^i(t)) &\sim (d_i(x), t) \\ (x, s^i(t)) &\sim (s_i(x), t). \end{aligned}$$

Example 2.5. Let X be a topological space. Consider the geometric realization $|S(X)|$ of the singular simplicial set of X. Define a continuous map $ev : |S(X)| \to X$ by $ev([f,t]) = f(t)$ for $(f,t) \in S_n(X) \times \Delta^n$. Then ev is a weak homotopy equivalence (i.e. ev induces isomorphisms on all homotopy groups) according to [13]. Hence when X is a connected CW complex, $|S(X)|$ and X are homotopy equivalent as a consequence of the Whitehead Theorem [9] since the geometric realization of a simplicial set is always a CW complex.

A simplicial map induces a continuous map between classifying spaces.

Definition 2.6. For a simplicial map $f : X \to Y$, define a continuous map $|f| : |X| \to |Y|$ by $|f|([(x,t)]) = [(f_n(x), t)]$ for $(x,t) \in X_n \times \Delta^n$.

We can construct a simplicial set from a small category.

Definition 2.7. For a small category C and a nonnegative number n, define $N_0(C) = C_0$ and for $n \neq 0$

$$N_n(C) = \{(f_n, \ldots, f_1) \in C_1^n \mid s(f_i) = t(f_{i-1})\}$$

Here C_0 is the set of objects in C, C_1 the set of morphisms, $s(f_i)$ is the domain of f_i and $t(f_{i-1})$ the range of f_{i-1}.

Definition 2.8. Let C be a small category. Define maps $d_i : N_n(C) \to N_{n-1}(C)$ by

$$d_i(f_n, f_{n-1}, \ldots, f_1) = \begin{cases} (f_n, f_{n-1}, \ldots, f_2) & (i = 0) \\ (f_n, f_{n-1}, \ldots, f_{i+2}, f_{i+1} \circ f_i, f_{i-1}, \ldots, f_1) & (1 \leq i \leq n-1) \\ (f_{n-1}, f_{n-2}, \ldots, f_1) & (i = n) \end{cases}$$

and maps $s_i : N_n(C) \to N_{n+1}(C)$ by

$$s_i(f_n, f_{n-1}, \ldots, f_1) = \begin{cases} (f_n, f_{n-1}, \ldots, f_1, id_{s(f_1)}) & (i = 0) \\ (f_n, f_{n-1}, \ldots, f_{i+1}, id_{t(f_i)}, f_i, \ldots, f_1) & (1 \leq i \leq n). \end{cases}$$

These maps satisfy the simplicial identities and one has the following well known fact.

Proposition 2.9. *Let C be a small category. Then $(N(C), d, s)$ is a simplicial set.*

Definition 2.10. Let C be a small category. The geometric realization $|N(C)|$ of $(N(C), d, s)$ is called the *classifying space* of C and is denoted by BC.

Definition 2.11. Let C and D be small categories. For a functor $F : C \to D$, define a simplicial map $N_n(F) : N_n(C) \to N_n(D)$ by

$$N_n(F)(x) = F_0(x) \quad , \text{ if } n = 0$$

and if $n \neq 0$,

$$N_n(F)(f_n, f_{n-1}, \ldots, f_1) = (F_1(f_n), F_1(f_{n-1}), \ldots, F_1(f_1))$$

Define then a (continuous) map $BF : BC \to BD$ by

$$BF([(x,t)]) = [(N_n(F)(x), t)].$$

Proposition 2.12. *Let C and D be small categories and $F : C \to D$ be a faithful functor. Suppose that the map $F_0 : C_0 \to D_0$ between the sets of objects is injective. Then $BF : BC \to BD$ is injective.*

Proof. For $[(x,t)] \in BC$, there are a natural number n and $(x', t') \in \overline{N}_n(C) \times \mathrm{Int}(\Delta^n)$ such that $(x,t) \sim (x', t')$ where

$$\begin{aligned} \overline{N}_n(C) = \{&(f_n, \ldots, f_1) \in N_n(C) : \\ &f_i \text{ is not the identity morphism for each } i\}. \end{aligned}$$

For $[(x,t)] \in BC$ and $[(y,s)] \in BD$ satisfying $(x,t) \in \overline{N}_n(C) \times \mathrm{Int}(\Delta^n)$ and $(y,s) \in \overline{N}_m(D) \times \mathrm{Int}(\Delta^m)$, assume that

$$BF([(x,t)]) = BF([(y,s)]).$$

Then $N_n(F)(x) \in \overline{N}_n(C)$ and $N_m(F)(y) \in \overline{N}_n(D)$ since F is faithful. Thus $(N_n(F)(x), t) \in \overline{N}_n(C) \times \mathrm{Int}(\Delta^n)$ and $(N_m(F)(y), s) \in \overline{N}_m(D) \times \mathrm{Int}(\Delta^m)$. Hence

$$\begin{aligned} (N_n(F)(x), t) = BF([(x,t)]) &= BF([(y,s)]) \\ &= (N_m(F)(y), s) \end{aligned}$$

and $(x,t) = (y,s)$ since F is faithful and F_0 is injective. □

3 Cellular stratified spaces

In this section, we review cellular stratified spaces introduced in [15] and [16].

Definition 3.1. Let X be a topological space. A subspace A of X is called *locally closed* if for any $x \in A$, there exists a neighborhood U_x of x such that $A \cap U_x$ is a closed set in U_x.

Definition 3.2. Let X be a topological space and Λ be a poset. A map $\pi : X \to \Lambda$ is called a *stratification* of X indexed by Λ if the map π satisfies the following conditions:

1. For $\lambda \in \operatorname{Im}\pi$, $\pi^{-1}(\lambda)$ is connected and locally closed.
2. For $\lambda, \lambda' \in \operatorname{Im}\pi$, $\pi^{-1}(\lambda) \subset \overline{\pi^{-1}(\lambda')}$ if and only if $\lambda \leq \lambda'$.

Definition 3.3. Let $\pi : X \to \Lambda$ be a stratification of a topological space X indexed by a poset Λ. For $\lambda \in \Lambda$, $\pi^{-1}(\lambda)$ is denoted by e_λ and is called *stratum* with index λ. $\operatorname{Im}\pi$ is denoted by $P(X)$ and is called the *face poset*. If $P(X)$ is finite, (X, π) is called *finite*.

Example 3.4. Define an order on $\{0, -1, 1\}$ by $0 < \pm 1$ and a map $\mathrm{sign} : \mathbb{R} \to \{0, -1, 1\}$ by

$$\mathrm{sign}(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x = 0) \\ -1 & (x < 0). \end{cases}$$

Then $(\mathbb{R}, \mathrm{sign})$ is a stratified space.

Example 3.5. Consider a hyperplane arrangement $\mathcal{A} = \{H_1, H_2, \ldots, H_k\}$ in $\mathbb{R}^n$ defined by affine maps $\{\ell_i : \mathbb{R}^n \to \mathbb{R}\}_{i=1}^k$. Define a map

$$\pi_{\mathcal{A}} : \mathbb{R}^n \to \{0, -1, 1\}^k$$

by $\pi_{\mathcal{A}}(x) = (\mathrm{sign}(\ell_1(x)), \mathrm{sign}(\ell_2(x)), \ldots, \mathrm{sign}(\ell_k(x)))$, where the set $\{0, -1, 1\}^k$ is equipped with the product order. Then $(\mathbb{R}^n, \pi_{\mathcal{A}})$ is a stratified space.

Definition 3.6. Let (X, π_X) and (Y, π_Y) be stratified spaces.

- A pair (f, s) of a continuous map $f : X \to Y$ and a poset map $s : P(X) \to P(Y)$ is called a *morphism of stratified spaces* if the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ P(X) & \xrightarrow{s} & P(Y) \end{array}$$

  and for each $\lambda \in P(X)$, $f(e_\lambda) = e_{s(\lambda)}$.

- When $X = Y$ and f is identity, (f, s) is called a *subdivision*.

Definition 3.7. Let (X, π) be a stratified space. For a stratum e_λ, $\overline{e_\lambda} - e_\lambda$ is called the *boundary* of e_λ and is denoted by ∂e_λ.

Definition 3.8. A stratified space (X, π) is called *CW* if it satisfies the following conditions:

1. For each stratum e_λ, the boundary ∂e_λ is covered by a finite number of strata;
2. X has the weak topology determined by the covering $\{\overline{e_\lambda} \mid \lambda \in P(X)\}$.

Definition 3.9. Let $\pi : X \to P(X)$ be a stratification on a Hausdorff space X.

- For a stratum e_λ, an *n-cell structure* on e_λ is a quotient map $\varphi_\lambda : D_\lambda \to X$ satisfying the following conditions:
  1. $\mathrm{Int}(D^n) \subset D_\lambda \subset D^n$
  2. $\varphi_\lambda(D_\lambda) = \overline{e_\lambda}$
  3. $\varphi_\lambda|_{\mathrm{Int}(D^n)} : \mathrm{Int}(D^n) \to e_\lambda$ is a homeomorphism.
- A stratum equipped with an n-cell structure is called an *n-cell*.
- The space D_λ is called the *domain* of e_λ.
- An n-cell e_λ is called *closed* if $D_\lambda = D^n$.

Example 3.10. Consider a stratification $(\mathbb{R}^n, \pi_{\mathcal{A}})$ in Example 3.5. When $\mathcal{A}$ is a central hyperplane arrangements (i.e. the affine map ℓ_i is a linear map for each i), we will define a cell structure on a stratum on $(\mathbb{R}^n, \pi_{\mathcal{A}})$ as follows.

Choose a stratum e in $(\mathbb{R}^n, \pi_{\mathcal{A}})$. Let

$$f : [0, 1) \to [0, \infty) \ , \ f(x) = \tan \frac{\pi}{2} x$$

and define

$$\psi : \overline{e} \cap \mathrm{Int}(D^n) \to \overline{e}$$

by

$$\psi(x) = \begin{cases} f(\|x\|) \frac{x}{\|x\|} & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

Then ψ is a continuous map since

$$\lim_{x \to 0} \|\psi(x)\| = \lim_{x \to 0} f(\|x\|) = 0.$$

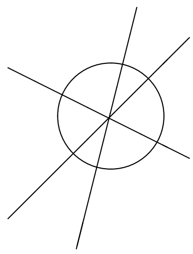
The inverse map to ψ is

$$\tilde{\psi} : \overline{e} \to \overline{e} \cap \mathrm{Int}(D^n)$$

defined by

$$\tilde{\psi}(x) = \begin{cases} f^{-1}(\|x\|) \frac{x}{\|x\|} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

Thus ψ is a homeomorphism. Hyperplanes cut the boundary ∂D^n of D^n and define a stratification $\pi_{\mathcal{A}, D^n}$ on D^n whose are all closed cells.



Thus $e \cap \mathrm{Int} D^n$ is a cell in D^n. Define a cell structure on e by the composition

$$D = \varphi^{-1}(\overline{e} \cap \mathrm{Int}(D^n)) \xrightarrow{\varphi} \overline{e} \cap \mathrm{Int}(D^n) \xrightarrow{\psi} \overline{e}$$

where $\varphi : D^m \to \overline{e \cap D^n}$ is the cell structure of $e \cap \mathrm{Int} D^n$ in D^n.

Definition 3.11. A *cellular stratification* on a stratified space (X, π) consists of a collection

$$\Phi = \{\varphi_\lambda : D_\lambda \to \overline{e_\lambda} \mid \lambda \in P(X)\}$$

of cell structures on strata satisfying the condition, that for each e_λ, ∂e_λ is covered by cells of dimension less than or equal to $n-1$

$$\partial e_\lambda \subset \bigcup_{\mu \in P(X), \dim e_\lambda \leq n-1} e_\mu.$$

The triple (X, π, Φ) is called a *cellular stratified space.*

Example 3.12. $\mathbb{R}^n$ is a cellular stratified space with the standard homeomorphism $\mathrm{Int}(D^n) \to \mathbb{R}^n$ as a cell structure of an n-cell. There are of course other distinct cellular stratification of $\mathbb{R}^n$ as illustrated by example 3.10.

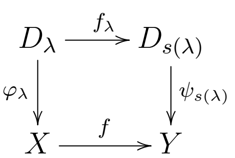
Definition 3.13. Let X be a cellular stratified space. A subset A of X is called a *cellular stratified subspace* of X if the following conditions are satisfied:

1. the restriction $\pi|_A : A \to \Lambda$ is a stratification.
2. for a cell e_λ in X satisfying $e_\lambda \subset A$, the restriction $\varphi_\lambda|_{D_{\lambda,A}} : D_{\lambda,A} = \varphi_\lambda^{-1}(\overline{e_\lambda} \cap A) \to \overline{e_\lambda} \cap A$ is a quotient map.

Definition 3.14. Let (X, π_X, Φ) and (Y, π_Y, Ψ) be cellular stratified spaces. A *morphism of cellular stratified spaces* consists of

- a morphism $(f, s) : (X, \pi_X) \to (Y, \pi_Y)$ of stratified spaces, and
- a family of maps $\{f_\lambda : D_\lambda \to D_{s(\lambda)}\}_{\lambda \in P(X)}$ indexed by cells $\varphi_\lambda : D_\lambda \to \overline{e_\lambda}$ in X

such that $f_\lambda(0) = 0$ for any $\lambda \in P(X)$ and the following diagram



is commutative, where $\psi_{s(\lambda)} : D_{s(\lambda)} \to Y$ is the cell structure of $e_{s(\lambda)}$ in Y.

4 Totally normal cellular stratified spaces and their barycentric subdivisions

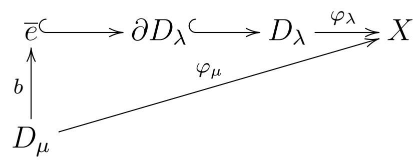
In this section, we define total normality for cellular stratified spaces and barycentric subdivisions of totally normal cellular stratified spaces.

Definition 4.1. Let X be a cellular stratified space.

- X is called *normal* if $e_\mu \cap \overline{e_\lambda} \neq \phi$ implies $e_\mu \subset \overline{e_\lambda}$ for any e_λ.
- X is called *regular* if the cell structure $\varphi : D_\lambda \to \overline{e_\lambda}$ of each e_λ is a homeomorphism onto $\overline{e_\lambda}$.

Definition 4.2. Let X be a normal cellular stratified space. Then X is called *totally normal* if for each n-cell e_λ,

1. there is a regular cell structure on S^{n-1} containing ∂D_λ as a cellular stratified subspace of S^{n-1},
2. for each cell e in the cellular stratification on ∂D_λ, there are a cell e_μ in X and a map $b : D_\mu \to \partial D_\lambda$ such that $b(D_\mu) = \overline{e}$, $b(\mathrm{Int}(D_\mu)) = e$, and the following diagram is commutative.

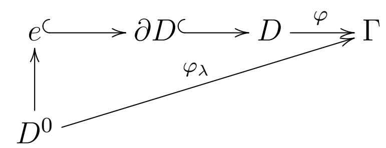


Definition 4.3. A 1-dimensional cellular stratified space is called a *graph.*

Graphs are examples of totally normal cellular stratified spaces.

Proposition 4.4. *Any graph is totally normal.*

Proof. Let Γ be a graph. Choose a 1-cell e with the cell structure $\varphi : D \to \overline{e}$ of e. The domain D of e is $(-1,1)$, $[-1,1)$, $(-1,1]$, or $[-1,1]$. When $\partial D = \phi$, the conditions of the definition of total normality are satisfied. When $\partial D \neq \phi$, ∂D is a stratified subspace of S^0 since ∂D is $\{1\}$, $\{-1\}$, or S^0. For a cell e in ∂D, there is a 0-cell e_λ in Γ such that the following diagram is commutative



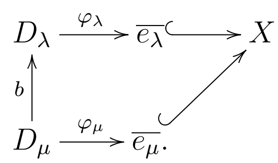
since the domain of e is D^0. ☐

We construct a small category from a totally normal cellular stratified space.

Definition 4.5. Let X be a totally normal cellular stratified space. Define a category $C(X)$ as follows. Objects are cells,

$$C(X)_0 = \{e \mid e \text{ is a cell in } X\}.$$

A morphism from a cell $\varphi_\mu : D_\mu \to \overline{e_\mu}$ to another cell $\varphi_\lambda : D_\lambda \to \overline{e_\lambda}$ is a map $b : D_\mu \to D_\lambda$ such that the following diagram is commutative

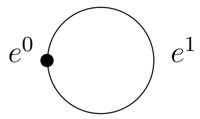


The composition is given by the composition of maps. This category $C(X)$ is called the *face category* of X.

Definition 4.6. The classifying space $BC(X)$ of the face category $C(X)$ of a totally normal cellular stratified space X is called the *barycentric subdivision* of X, and is denoted by $\mathrm{Sd}\,(X)$.

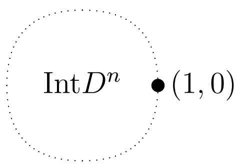
Example 4.7. $\mathbb{R}^n$ is a totally normal cellular stratified space by Example 3.12. The barycentric subdivision $\mathrm{Sd}\,(\mathbb{R}^n)$ of $\mathbb{R}^n$ is a single point.

Example 4.8. Consider $S^1 = e^0 \cup e^1$ where e^0 is a point in S^1 and $e^1 = S^1 - e^0$.



S^1 is totally normal since S^1 is a graph. The barycentric subdivision $\mathrm{Sd}\,(S^1)$ is the boundary of a hexagon and is homeomorphic to S^1.

Example 4.9. Consider $X = \mathrm{Int}D^n \cup \{(1,0)\}$ consisting of an n-cell and a 0-cell.



Here X is totally normal since the boundary ∂e^n of the n-cell $e^n = \mathrm{Int}(D^n)$ is a point. The barycentric subdivision $\mathrm{Sd}\,(X)$ is a 1-simplex.

The barycentric subdivision $\mathrm{Sd}\,(X)$ of a CW totally normal cellular stratified space X is embedded into X.

Theorem 4.10 ([15])**.** *For a CW totally normal cellular stratified space X, there is a simplicial map $i : N(C(X)) \to S(X)$ from the nerve $N(C(X))$ of the face category $C(X)$ to the singular simplicial set $S(X)$ of X such that the composition*

$$\tilde{i} : \mathrm{Sd}\,(X) \xrightarrow{|i|} |S(X)| \xrightarrow{ev} X$$

is an embedding, where ev is a continuous map defined in Example 2.5.

The following theorem is the key result to study configuration spaces of graphs in this paper.

Theorem 4.11 ([15])**.** *For a CW totally normal cellular stratified space X, the image $\tilde{i}(\mathrm{Sd}\,(X))$ of the embedding $\tilde{i} : \mathrm{Sd}\,(X) \to X$ is a strong deformation retract of X.*

Proposition 4.12. *For a totally normal cellular stratified space X, $\mathrm{Sd}\,(X)$ is a regular cell complex.*

Proof. It is obvious by the definition of the geometric realization. □

5 Subspaces, Products, and Subdivisions

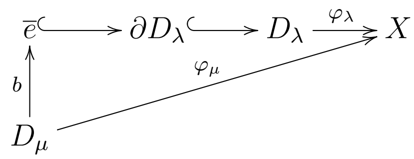
In this section, we prove total normality for cellular stratified subspaces, product cellular stratified spaces, and cellular subdivisions to study configuration spaces of graphs.

First, let us consider subspaces.

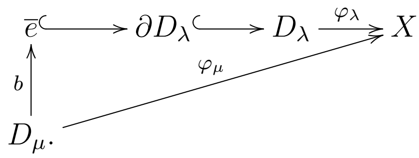
Theorem 5.1. *Let X be a totally normal cellular stratified space. A cellular stratified subspace A of X is totally normal if for any $\lambda \in P(A)$, $\partial D_{\lambda,A}$ is a cellular stratified subspace of ∂D_λ where $D_{\lambda,A}$ is defined by Definition 3.13.*

Proof. For any $\lambda \in P(A)$,

1. there is a regular cell structure on $S^{\dim e_\lambda - 1}$ containing ∂D_λ as a cellular stratified subspace of $S^{\dim e_\lambda - 1}$, and
2. for each cell e in the cellular stratification on ∂D_λ, there are a cell e_μ in X and a map $b : D_\mu \to \partial D_\lambda$ such that $b(D_\mu) = \overline{e}$, $b(\mathrm{Int}(D_\mu)) = e$, and the following diagram is commutative



since X is totally normal. A cell in $\partial D_{\lambda,A}$ is a cell in ∂D_λ since $\partial D_{\lambda,A}$ is a stratified subspace of ∂D_λ. Then given a cell e in $\partial D_{\lambda,A}$, there are a cell e_μ in X and a map $b : D_\mu \to \partial D_\lambda$ such that $b(D_\mu) = \overline{e}$, $b(\mathrm{Int}(D_\mu)) = e$, and the following diagram is commutative



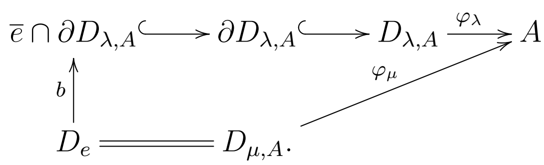
We have $D_e = b^{-1}(\overline{e} \cap \partial D_{\lambda,A}) \subset D_{\mu,A}$ since for $x \in D_e$,

$$\varphi_\mu(x) = \varphi_\lambda(b(x)) \in A.$$

Conversely, $D_{\mu,A}$ is included in D_e since for $x \in D_{\mu,A}$,

$$\varphi_\lambda(b(x)) = \varphi_\mu(x) \in \overline{e_\mu} \cap A \subset A,$$

then $b(x) \in \overline{e} \cap \partial D_{\lambda,A}$. So $D_e = D_{\mu,A}$. Then the following diagram is commutative



We also have $e_\mu \subset A$ since for $x \in \mathrm{Int}(D_\mu)$,

$$\varphi_\lambda(b(x)) \in \varphi_\lambda(e) \subset \varphi_\lambda(\partial D_{\lambda,A}) \subset \overline{e_\lambda} \cap A \subset A.$$

Thus A is totally normal. □

Second, let us consider products.

Definition 5.2. Let P and Q be posets. Define a partial order on $P \times Q$ as follows: for (x, y) and (x', y') in $P \times Q$,

$$(x, y) \leq (x', y') \text{ if only if } x \leq x' \text{ and } y \leq y'$$

Proposition 5.3 ([15])**.** *Let (X, π_X) and (Y, π_Y) be stratified spaces. Define $P(X \times Y) = P(X) \times P(Y)$. Then the product $(X \times Y, \pi_X \times \pi_Y)$ is a stratified space.*

Definition 5.4. Let X and Y be cellular stratified spaces. For cells $e_\lambda \subset X$ and $e_\mu \subset Y$, consider the composition

$$\varphi_{\lambda,\mu} : D_{\lambda,\mu} \cong D_\lambda \times D_\mu \xrightarrow{\varphi_\lambda \times \varphi_\mu} \overline{e_\lambda} \times \overline{e_\mu} = \overline{e_\lambda \times e_\mu}$$

mapping into $X \times Y$, where $D_{\lambda,\mu}$ is the subspace of $D^{\dim e_\lambda + \dim e_\mu}$ obtained by pulling back $D_\lambda \times D_\mu$ via the standard homeomorphism

$$D^{\dim e_\lambda + \dim e_\mu} \cong D^{\dim e_\lambda} \times D^{\dim e_\mu}.$$

$X \times Y$ is called the *product cellular stratified space* if $\varphi_{\lambda,\mu}$ is a quotient map for each pair of cells in X and Y.

In general, $f \times g : X_1 \times Y_1 \to X_2 \times Y_2$ is not necessarily quotient even if $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ are quotient maps. Bi-quotient maps are useful to solve this problem.

Definition 5.5. Let $f : X \to Y$ be surjective and continuous. Then f is called *bi-quotient* if for any $y \in Y$ and any open covering $\mathcal{U}$ of $f^{-1}(y)$, there exist open sets $U_1, U_2, \ldots, U_k \in \mathcal{U}$ such that $\bigcup_{i=1}^{k} f(U_i)$ contains a neighborhood of y.

Lemma 5.6 ([12])**.** *Let $\{f_i : X_i \to Y_i\}_{i \in I}$ be a family of bi-quotient maps indexed by a set I. Then the product $\Pi_{i \in I} f_i : \Pi_{i \in I} X_i \to \Pi_{i \in I} Y_i$ is a quotient map.*

The problem is when a characteristic map of a cell is bi-quotient.

Definition 5.7. Let $f : X \to Y$ be a continuous map. The map f is called *relatively compact* if $f^{-1}(y)$ is compact for each $y \in Y$.

Lemma 5.8 ([16])**.** *Let X be a cellular stratified space. If the cell structure $\varphi_\lambda : D_\lambda \to \overline{e_\lambda}$ of e_λ is relatively compact for any $\lambda \in P(X)$, then φ_λ is bi-quotient.*

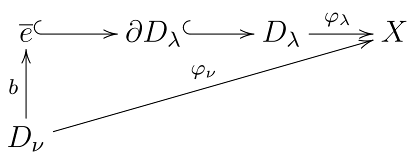
For a totally normal cellular stratified space, any cell structures are bi-quotient.

Lemma 5.9. *Let X be a totally normal cellular stratified space. Then for any $\lambda \in P(X)$, the cell structure $\varphi_\lambda : D_\lambda \to \overline{e_\lambda}$ of e_λ is bi-quotient.*

Proof. For any $y \in \partial e_\lambda$, there is $\mu \in P(X)$ such that $y \in e_\mu$ and $\dim e_\mu \leq \dim e_\lambda - 1$ since X is cellular. Also $e_\mu \subset \overline{e_\lambda}$ since X is normal. Define

$$f : C(X)(e_\mu, e_\lambda) \times \{y\} \to \varphi_\lambda^{-1}(y)$$

by $f(b, y) = b(\varphi_\mu^{-1}(y))$. Here f is continuous since $C(X)(e_\mu, e_\lambda) \times \{y\}$ is a discrete space. For any $x \in \varphi_\lambda^{-1}(y)$, there is a cell e in ∂D_λ such that $x \in e$. There are a cell e_ν in X and a map $b : D_\nu \to \partial D_\lambda$ such that the following diagram



is commutative since X is totally normal.

There is $z \in \mathrm{Int}(D_\nu)$ such that $b(z) = x$ since the restriction $b|_{\mathrm{Int}(D_\nu)} : \mathrm{Int}(D_\nu) \to e$ is a homeomorphism. $\mu = \nu$ since

$$y = \varphi_\lambda(x) = \varphi_\lambda(b(z)) = \varphi_\nu(z) \in e_\nu \cap e_\mu.$$

Then f is surjective since $b \in C(X)_1(e_\mu, e_\lambda)$ and

$$f(b, y) = b(\varphi_\mu^{-1}(y)) = b(z) = x.$$

$C(X)(e_\mu, e_\lambda) \times \{y\}$ is compact since $C(X)(e_\mu, e_\lambda)$ is finite. Then $\varphi_\lambda^{-1}(y)$ is compact. Thus φ_λ is bi-quotient by Lemma 5.8. $\square$

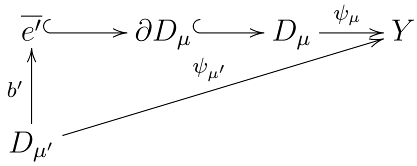
The product of totally normal cellular stratified spaces is totally normal.

Theorem 5.10. *Let X and Y be totally normal cellular stratified spaces. Then the product $X \times Y$ is totally normal.*

Proof. For $\lambda \in P(X)$ and $\mu \in P(Y)$, the product $\varphi_\lambda \times \psi_\mu$ of cell structures φ_λ and ψ_μ of e_λ and e_μ is a quotient map by Lemma 5.6 and 5.9. Thus $X \times Y$ is a cellular stratified space. Assume that e_λ is an n-cell and e_μ is an m-cell. There are regular cell structures on S^{n-1} and S^{m-1} containing ∂D_λ and ∂D_μ as cellular stratified subspaces of S^{n-1} and S^{m-1} respectively since X and Y are totally normal.

The standard homeomorphism $D^{n+m} \cong D^n \times D^m$ induces the homeomorphism $S^{n+m-1} \cong (D^n \times S^{m-1}) \cup (S^{n-1} \times D^m)$. The cell structures on D^n and D^m are defined by adding an n-cell and an m-cell to the regular cell structures of S^{n-1} and S^{m-1}, respectively. Then we obtain the cell structure on $(D^n \times S^{m-1}) \cup (S^{n-1} \times D^m)$. This cellular stratification contains $\partial D_{\lambda,\mu} \cong (D_\lambda \times \partial D_\mu) \cup (\partial D_\lambda \times D_\mu)$ as a stratified subspace. Choose a cell $e \times e' \subset D_\lambda \times \partial D_\mu$.

1. When e is an n-cell, the domain of e is D_λ. There are a cell $e_{\mu'}$ in Y and a map $b' : D_{\mu'} \to \partial D_\mu$ such that the following diagram is commutative.



Then the following diagram is commutative.

$$\begin{array}{ccccc} \overline{e} \times \overline{e'} & \hookrightarrow & D_\lambda \times \partial D_\mu & \hookrightarrow & D_\lambda \times D_\mu \\ \uparrow {\scriptstyle id_{D_\lambda} \times b'} & & & & \downarrow {\scriptstyle \varphi_\lambda \times \psi_\mu} \\ D_\lambda \times D_{\mu'} & & \xrightarrow{\varphi_\lambda \times \psi_{\mu'}} & & X \times Y \end{array}$$

2. When e is a cell in ∂D_λ, there are a cell $e_{\lambda'}$ in X and a map $b : D_{\lambda'} \to \partial D_\lambda$ such that the following diagram is commutative.

$$\begin{array}{ccccccc} \overline{e} & \hookrightarrow & \partial D_\lambda & \hookrightarrow & D_\lambda & \xrightarrow{\varphi_\lambda} & X \\ \uparrow {\scriptstyle b} & & & & & \nearrow {\scriptstyle \varphi_{\lambda'}} & \\ D_{\lambda'} & & & & & & \end{array}$$

Then the following diagram is commutative.

$$\begin{array}{ccccc} \overline{e} \times \overline{e'} & \hookrightarrow & \partial D_\lambda \times \partial D_\mu & \hookrightarrow & D_\lambda \times D_\mu \\ \uparrow {\scriptstyle b \times b'} & & & & \downarrow {\scriptstyle \varphi_\lambda \times \psi_\mu} \\ D_{\lambda'} \times D_{\mu'} & & \xrightarrow{\varphi_{\lambda'} \times \psi_{\mu'}} & & X \times Y \end{array}$$

Thus $X \times Y$ is totally normal. □

Third, let us consider cellular subdivision.

Definition 5.11. Let (X, π, Φ) be a cellular stratified space. A *cellular subdivision* of (π, Φ) consists of

- a subdivision of stratified spaces

$$(id_X, s) : (X, \pi') \to (X, \pi),$$

- a regular cellular stratification $(\pi_\lambda, \Phi_\lambda)$ on the domain D_λ of each cell e_λ in (π, Φ)

satisfying the following conditions:

1. $\mathrm{Int}(D_\lambda)$ is a stratified subspace of $(D_\lambda, \pi_\lambda, \Phi_\lambda)$.
2. For each $\lambda \in P(X)$, the cell structure $\varphi_\lambda : (D_\lambda, \pi_\lambda) \to (X, \pi')$ of e_λ is a morphism of stratified spaces.
3. The map $P(\varphi_\lambda) : P(\mathrm{Int}(D_\lambda)) \to P(e_\lambda)$ induced by the cell structure φ_λ is a bijection.

We will define a cellular subdivision of the n-fold product Γ^n of a graph Γ by defining subdivisions of domains of cell structures in order to obtain a totally normal cellular stratification on configuration spaces.

Theorem 5.12 ([16])**.** *Let X be a totally normal cellular stratified space and $(D_\lambda, \pi_\lambda, \Phi_\lambda)$ be a regular cellular stratification for each $\lambda \in P(X)$. Assume that each morphism $b : (D_\mu, \pi_\mu, \Phi_\mu) \to (D_\lambda, \pi_\lambda, \Phi_\lambda)$ in the face category $C(X)$ of (X, π, Φ) is a morphism of stratified spaces. Define a stratification π' on X by*

$$\pi' : X = \bigcup_{\lambda \in P(X,\pi,\Phi)} \bigcup_{\lambda' \in P(\mathrm{Int}(D_\lambda))} \varphi_\lambda(e_{\lambda'}).$$

Then $((X, \pi'), (D_\lambda, \pi_\lambda, \Phi_\lambda))$ is a cellular subdivision.

The restriction of f to a subspace A is not necessarily quotient even if $f : X \to Y$ is a quotient map. Hereditarily quotient maps are useful to solve this problem.

Definition 5.13. Let $f : X \to Y$ be a surjective and continuous map from a topological space X to a topological space Y. The map f is called *hereditarily quotient* if for $y \in Y$ and a neighborhood U of $f^{-1}(y)$, $f(U)$ is a neighborhood of y.

Lemma 5.14 ([3])**.** *Let $f : X \to Y$ be hereditarily quotient.*

1. *f is a quotient.*
2. *For a subspace A of Y, the restriction*

$$f|_{f^{-1}(A)} : f^{-1}(A) \to A$$

is hereditarily quotient.

For a cellular stratified space X, We define a new cellular stratification on X obtained by a cellular subdivision of X.

Definition 5.15. Let (X, π, Φ) be a cellular stratified space whose cell structures are hereditarily quotient and let $((X, \pi'), (D_\lambda, \pi_\lambda, \Phi_\lambda))$ be a cellular subdivision of (X, π, Φ). Define a cellular stratification on (X, π') as follows.

For $\lambda \in P(X, \pi)$ and $\lambda' \in P(\mathrm{Int}(D_\lambda))$, define a cell structure of $\pi'^{-1}(P(\varphi_\lambda)(\lambda'))$ by the composition

$$D_{\lambda'} \xrightarrow{s_{\lambda'}} D_\lambda \xrightarrow{\varphi_\lambda} X$$

where $s_{\lambda'} : D_{\lambda'} \to D_\lambda$ is the cell structure of the cell $e_{\lambda'}$ in $(D_\lambda, \pi_\lambda, \Phi_\lambda)$.

Theorem 5.16 ([5])**.** *Let (X, π, Φ) be a totally normal cellular stratified space whose cell structures are hereditarily quotient and let $((X, \pi'), (D_\lambda, \pi_\lambda, \Phi_\lambda))$ be a cellular subdivision of (X, π, Φ). Assume that each morphism $b : (D_\mu, \pi_\mu, \Phi_\mu) \to (D_\lambda, \pi_\lambda, \Phi_\lambda)$ in the face category $C(X)$ of (X, π, Φ) is a morphism of stratified spaces. Then*

$$\pi' : X = \bigcup_{\lambda \in P(X,\pi,\Phi)} \bigcup_{\lambda' \in P(\mathrm{Int}(D_\lambda))} \varphi_\lambda(e_{\lambda'})$$

is totally normal.

6 Totally normal cellular stratifications for configuration spaces on graphs

We construct a stratification on the configuration space $C_n(\Gamma)$ of a graph Γ. First, we construct subdivisions of the domains of cell structures of cells in Γ^n.

Definition 6.1. For $1 \leq i < j \leq n$, define a hyperplane $H_{i,j}$ by

$$H_{i,j} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}.$$

and a hyperplane arrangement $\mathcal{A}_{n-1}$ in $\mathbb{R}^n$ by

$$\mathcal{A}_{n-1} = \{H_{i,j} \mid 1 \le i < j \le n\}.$$

This hyperplane arrangement $\mathcal{A}_{n-1}$ is called the *braid arrangement*.

A cell in $(\mathbb{R}^n, \pi_{\mathcal{A}_{n-1}})$ is described by an ordered partition.

Definition 6.2. Let X be a finite set. A sequence $\{s_1, s_2, \ldots, s_k\}$ of subsets of X is called an *ordered partition* of X if the following conditions satisfy:

- $s_i \neq \phi$ for each i
- $X = \coprod_{i=1}^k s_i$.

An ordered partition $\{s_1, s_2, \ldots, s_k\}$ of $\{1, 2, \ldots, n\}$ is denoted by $(s_1|s_2|\ldots|s_k)$.

Definition 6.3. For an ordered set partition $\tau = (s_1|s_2|\ldots|s_k)$ of $\{1, 2, \ldots, n\}$, a cell $e(\tau)$ in $(\mathbb{R}^n, \pi_{\mathcal{A}_{n-1}})$ is defined by

$$\begin{aligned} e(\tau) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid\ & x_{i_1^{(1)}} = x_{i_2^{(1)}} = \ldots = x_{i_{\ell_1}^{(1)}} \\ & < x_{i_1^{(2)}} = \ldots = x_{i_{\ell_2}^{(2)}} < \\ & \ldots < x_{i_1^{(k)}} = \ldots = x_{i_{\ell_k}^{(k)}}\} \end{aligned}$$

where $s_i = \{i_1^{(i)}, i_2^{(i)}, \ldots, i_{\ell_i}^{(i)}\}$.

Definition 6.4. Let $X = (-1,1), [-1,1), (-1,1]$, or $[-1,1]$. Define a regular cellular stratification on a stratified subspace $(X^n, \pi_{\mathcal{A}_{n-1}}|_{X^n})$ of $(\mathbb{R}^n, \pi_{\mathcal{A}_{n-1}})$ as follows.

For a stratum e in X^n, let D be the closure of e in X^n. Define the inclusion $D \hookrightarrow X^n$ as a cell structure of e.

This cellular stratification is denoted by $\Phi(X, n)$.

Definition 6.5. Let X be a totally normal cellular stratified space. For $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in P(X^n)$,

- A cell $e_{\lambda_1} \times e_{\lambda_2} \times \ldots \times e_{\lambda_n}$ in X^n is denoted by $e_{\boldsymbol{\lambda}}$.
- The cell structure $\varphi_{\lambda_1} \times \ldots \times \varphi_{\lambda_n}$ is denoted by $\varphi_{\boldsymbol{\lambda}}$.
- The domain $D_{\lambda_1} \times D_{\lambda_2} \times \ldots \times D_{\lambda_n}$ is denoted by $D_{\boldsymbol{\lambda}}$.

Definition 6.6. Let Γ be a finite graph. We choose a total order on the set of 1-cells. For a cell $e_{\boldsymbol{\lambda}}$ in Γ^n, define a regular cellular stratification on $D_{\boldsymbol{\lambda}}$ as follows.

There is a permutation $\sigma \in \Sigma_n$ such that

$$\begin{aligned}(e_{\boldsymbol{\lambda}})\sigma = \{\text{a product of 0-cells}\} \times (e^1_{\mu_1})^{m_1} \times (e^1_{\mu_2})^{m_2} \\ \times \ldots \times (e^1_{\mu_k})^{m_k}\end{aligned}$$

and $\mu_1 < \mu_2 < \ldots < \mu_k$. Define a cellular stratification on $(D_{\mu_1})^{m_1} \times (D_{\mu_2})^{m_2} \times \ldots \times (D_{\mu_k})^{m_k}$ by the product cellular stratification of

$$\{((D_{\mu_i})^{m_i}, \pi_{\mathcal{A}_{m_i-1}}|_{(D_{\mu_i})^{m_i}}, \Phi(D_{\mu_i}, m_i))\}.$$

For a cell e in $(D_{\mu_1})^{m_1} \times (D_{\mu_2})^{m_2} \times \ldots \times (D_{\mu_k})^{m_k}$, $(e)\sigma^{-1}$ defines a cell in $D_{\boldsymbol{\lambda}}$. These cells define a stratification on $D_{\boldsymbol{\lambda}}$. This stratification is denoted by $\pi_{\boldsymbol{\lambda}}$.

Theorem 6.7. *Let Γ be a finite graph. Let $\tilde{\pi}$ be a stratification on Γ^n defined by Theorem 5.12. Then $(\Gamma^n, \tilde{\pi})$ is a totally normal cellular stratified space.*

Proof. The product Γ^n is totally normal by Proposition 4.4 and Theorem 5.10.

We will prove that the cell structure $\varphi_\lambda : D_\lambda \to \overline{e}$ of a 1-cell e in Γ is hereditarily quotient. If e is not a loop, φ is hereditarily quotient since φ is a homeomorphism. We prove the case when e is a loop. Choose a point $y \in e$ and a neighborhood U of $\varphi^{-1}(y)$. Then there is an open set V in $(-1,1)$ such that

$$\varphi^{-1}(y) \in V \subset U.$$

Thus $\varphi(U)$ is a neighborhood of y since the restriction $\varphi|_{(-1,1)}$ is a homeomorphism. Choose the point $y \in \partial e$ and a neighborhood U of $\varphi^{-1}(y)$. The neighborhood U decomposes into a neighborhood U_1 of 1 and a neighborhood U_2 of -1. Thus $\varphi(U)$ is a neighborhood of y. Hence φ is hereditarily quotient.

For a morphism

$$b : (D_{\boldsymbol{\mu}}, \pi_{\boldsymbol{\mu}}) \to (D_{\boldsymbol{\lambda}}, \pi_{\boldsymbol{\lambda}})$$

in the face category of Γ^n, b is a morphism of stratified spaces by Proposition 3.7 in [5].

Thus $((\Gamma^n, \tilde{\pi}), (D_{\boldsymbol{\lambda}}, \pi_{\boldsymbol{\lambda}}))$ is a cellular subdivision by Theorem 5.12 and $(\Gamma^n, \tilde{\pi})$ is totally normal by Theorem 5.16. ☐

The configuration space $C_n(\Gamma)$ is contained in $(\Gamma^n, \tilde{\pi})$ as a cellular stratified subspace. In order to show that $C_n(\Gamma)$ is totally normal, it suffices to verity the condition in Theorem 5.1.

Theorem 6.8. *Let (Γ, π) be a finite graph. Then the configuration space $C_n(\Gamma)$ of Γ has a totally normal cellular stratification.*

Proof. For a stratum e in $C_n(\Gamma)$, there are $\boldsymbol{\lambda} \in P(\Gamma^n, \pi^n)$ and $\nu \in P(\mathrm{Int}(D_{\boldsymbol{\lambda}}))$ such that

$$e = (\varphi_{\boldsymbol{\lambda}})(e_\nu)$$

There is a permutation $\sigma \in \Sigma_n$ such that

$$(e_{\boldsymbol{\lambda}})\sigma = \{\text{a product of 0-cells}\} \times (e^1_{\mu_1})^{m_1} \times \ldots \times (e^1_{\mu_k})^{m_k}$$

and $\mu_1 < \mu_2 < \ldots < \mu_k$. For each $1 \le i \le k$, there is a cell e_i in $((D_{\mu_i})^{m_i}, \pi_{\mathcal{A}_{m_i-1}})$ such that

$$e_\nu = (\{\text{a product of domains of 0-cells}\} \times e_1 \times \ldots \times e_k)\sigma^{-1}.$$

Define a subspace D_i of $(D_{\mu_i})^{m_i}$ by

$$D_i = \text{The closure of } e_i \text{ in } (D_{\mu_i})^{m_i}.$$

Then $\tilde{D} = \{\text{a product of domains of 0-cells}\} \times D_1 \times \ldots \times D_k$ is the domain of e in $(\Gamma^n, \tilde{\pi})$. The domain $(\tilde{D})_{C_n(\Gamma)}$ of e in $C_n(\Gamma)$ is obtained by removing some faces from $\tilde{D}$. Thus $\partial(\tilde{D})_{C_n(\Gamma)}$ is a stratified subspace of $\partial(\tilde{D})$. Hence $C_n(\Gamma)$ has a totally normal cellular stratification by Theorem 5.1. ☐

Let us introduce notations for cells in $C_n(\Gamma)$ based on the correspondence in Definition 6.3.

Definition 6.9. Let Γ be a graph. For a cell e in $C_n(\Gamma)$, there are a cell $e_{\boldsymbol{\lambda}}$ in Γ^n and a cell e_ν in $\mathrm{Int}(D_{\boldsymbol{\lambda}})$ such that

$$e = (\varphi_{\boldsymbol{\lambda}})(e_\nu).$$

There is a permutation $\sigma \in \Sigma_n$ such that

$$(e_{\boldsymbol{\lambda}})\sigma = \{\text{a product of 0-cells}\} \times (e^1_{\mu_1})^{m_1} \times (e^1_{\mu_2})^{m_2} \times \ldots \times (e^1_{\mu_k})^{m_k}$$

and $\mu_1 < \mu_2 < \ldots < \mu_k$.

For each $1 \le i \le k$, there is a cell e_i in

$$((D_{\mu_i})^{m_i}, \pi_{\mathcal{A}_{m_i-1}}, \Phi(D_{\mu_i}, m_i))$$

such that

$$e_\nu = (\{\text{a product of 0-cells}\} \times e_1 \times e_2 \times \ldots \times e_k)\sigma^{-1}.$$

For each $1 \le i \le k$, Let τ_i be the partition of A_i corresponding to e_i under the one-to-one correspondence in Definition 6.3, where

$$A_i = \left\{ x \in \{1, 2, \ldots, n\} \;\middle|\; e_{\lambda_x} = e^1_{\mu_i} \right\}.$$

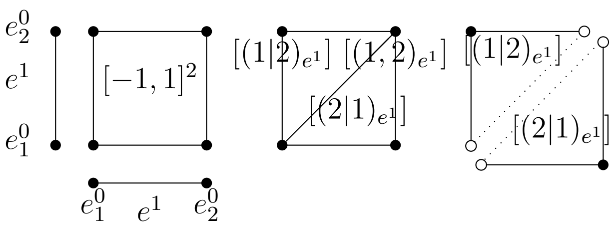
In this case, the cell e is denoted by $[\tau_{1\mu_1}, \tau_{2\mu_2}, \ldots, \tau_{k\mu_k}]$.

Example 6.10. We consider a cellular stratification on $C_n(\Gamma)$ and $\mathrm{Sd}\,(C_n(\Gamma))$ when $\Gamma = [-1, 1]$ and $n = 2$. The interval $[-1, 1]$ has a stratification: $[-1.1] = e^0_1 \cup e^0_2 \cup e^1$ where $e^0_1 = \{-1\}, e^0_2 = \{1\}$, and $e^1 = (-1, 1)$. The product stratification $[-1, 1]^2$ is given by

$$[-1, 1]^2 = e^0_1 \times e^0_1 \cup e^0_1 \times e^1 \cup e^0_1 \times e^0_2 \cup e^1 \times e^0_1 \cup e^1 \times e^1 \cup e^1 \times e^0_2 \cup e^0_2 \times e^0_1 \cup e^0_2 \times e^1 \cup e^0_2 \times e^0_2.$$

The diagonal set $H_{1,2} = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$ subdivides the square $[-1, 1]^2$. The 2-cell $e^1 \times e^1$ is divided into a 1-cell $[(1, 2)_{e^1}]$ and two 2-cells $[(1|2)_{e^1}]$ and $[(2|1)_{e^1}]$. We obtain a cellular stratification on $C_2([-1, 1])$ by removing from $([-1, 1]^2, \pi_{\mathcal{A}_1}|_{[-1,1]^2})$ the subspace $\overline{[(1, 2)_{e^1}]}$.

The barycentric subdivision $\mathrm{Sd}\,(C_2([-1, 1]))$ is homotopy equivalent to S^0.



7 Connectivity of configuration spaces of graphs

In this section, we prove Theorem 1.5.

A path in a regular cell complex is described by a sequence of 1-cells.

Definition 7.1. Let X be a regular cell complex. For two 0-cells u, v, a *path* from u to v is a sequence $P = (e_1, e_2, \ldots, e_n)$ of 1-cells such that $\overline{e_i} \cap \overline{e_{i+1}}$ is a 0-cell for each i, $\overline{e_{i-1}} \cap \overline{e_i}$ and $\overline{e_i} \cap \overline{e_{i+1}}$ are distinct 0-cells for each i, $u \in \partial e_1 - \partial e_2$, and $v \in \partial e_n - \partial e_{n-1}$.

We can remove all leaves from a given graph without changing homotopy type of the configuration space.

Lemma 7.2 ([5])**.** *Let Γ be a graph. Define a subgraph Γ° of Γ by removing all leaves from Γ. Then the inclusion $\Gamma^\circ \hookrightarrow \Gamma$ induces a homotopy equivalence $C_n(\Gamma^\circ) \simeq C_n(\Gamma)$.*

We can assume that for any natural numbers k and ℓ, $\Gamma_{k,\ell}$ has only one vertex by Lemma 7.2 and the domains of all branches are $(-1, 1]$.

Lemma 7.3. *For $k \ge 3$, $C_n(\Gamma_{k,0})$ is path connected.*

Proof. It suffices to prove that $\mathrm{Sd}\,(C_n(\Gamma_{k,0}))$ is path connected, by Theorem 4.11. We will prove that for any two vertices v, w in $\mathrm{Sd}\,(C_n(\Gamma_{k,0}))$, there is a path from v to w in $\mathrm{Sd}\,(C_n(\Gamma_{k,0}))$. Let $\Gamma_{k,0} = e^0 \cup \bigcup_{i=1}^k e^1_{\lambda_i}$ be a stratification on $\Gamma_{k,0}$. For two cells $[\sigma_{1\lambda_1}, \ldots, \sigma_{k\lambda_k}]$ and $[\rho_{1\lambda_1}, \ldots, \rho_{k\lambda_k}]$ in $C_n(\Gamma_{k,0})$, there are numbers $x \in \{1, 2, \ldots, n\}$ and $i \in \{1, 2, \ldots, k\}$ such that

$$[\sigma_{1\lambda_1}, \ldots, (\sigma_i|x)_{\lambda_i}, \ldots, \sigma_{k\lambda_k}] = [\rho_{1\lambda_1}, \ldots, \rho_{k\lambda_k}]$$

if and only if there is a morphism from $[\sigma_{1\lambda_1}, \ldots, \sigma_{k\lambda_k}]$ to $[\rho_{1\lambda_1}, \ldots, \rho_{k\lambda_k}]$ in the face category $C(C_n(\Gamma_{k,0}))$ by Definition 6.9. For two numbers $i, j \in \{1, 2, \ldots, k\}$, there is a number $h \in \{1, 2, \ldots, k\}$ such that h is not i nor j since $k \ge 3$. We may assume without a loss of generality that $i < h < j$. Hence for two cells
$\alpha = [\sigma_{1\lambda_1}, \ldots, (\tau_1|x|a_1|\ldots|a_p)_{\lambda_i}, \ldots, \sigma_{k\lambda_k}]$ and
$\beta = [\sigma_{1\lambda_1}, \ldots, (\tau_2|x|b_1|\ldots|b_q)_{\lambda_j} \ldots, \sigma_{k\lambda_k}]$ in $C_n(\Gamma_{k,0})$ satisfying $(\tau_1|a_1|\ldots|a_p) = \sigma_i$ and $(\tau_2|b_1|\ldots|b_q) = \sigma_j$, a path from α to β is obtained by the following sequence:

$$\alpha \leftarrow [\sigma_{1\lambda_1}, \ldots, (\tau_1|x|a_1|\ldots|a_{p-1})_{\lambda_i} \ldots, \sigma_{k\lambda_k}] \to [\sigma_{1\lambda_1} \ldots, (\tau_1|x|a_1|\ldots|a_{p-1})_{\lambda_i}, \ldots, (\sigma_h|a_p)_{\lambda_h}, \ldots, \sigma_{k\lambda_k}]$$

$$\leftarrow \ldots \to [\sigma_{1\lambda_1}, \ldots, (\tau_1|x)_{\lambda_i}, \ldots, (\sigma_h|a_p|\ldots|a_1)_{\lambda_h}, \ldots, \sigma_{k\lambda_k}]$$

$$\leftarrow [\sigma_{1\lambda_1}, \ldots, (\tau_1|x)_{\lambda_i}, \ldots, (\sigma_h|a_p|\ldots|a_1)_{\lambda_h}, \ldots, (\tau_2|b_1|\ldots|b_{q-1})_{\lambda_j}, \ldots, \sigma_{k\lambda_k}]$$

$$\to [\sigma_{1\lambda_1}, \ldots, (\tau_1|x)_{\lambda_i}, \ldots, (\sigma_h|a_p|\ldots|a_1|b_q)_{\lambda_h}, \ldots, (\tau_2|b_1|\ldots|b_{q-1})_{\lambda_j}, \ldots, \sigma_{k\lambda_k}]$$

$$\leftarrow \dots \to [\sigma_{1\lambda_1}, \dots, (\tau_1|x)_{\lambda_i}, \dots, (\sigma_h|a_p|\dots|a_1|b_q|\dots |b_1)_{\lambda_h}, \dots, \tau_{2\lambda_j}, \dots, \sigma_{k\lambda_k}]$$

$$\leftarrow \dots \to [\sigma_{1\lambda_1}, \dots, \tau_{1\lambda_i}, \dots, (\sigma_h|a_p|\dots|a_1|b_q|\dots|b_1)_{\lambda_h}, \dots, (\tau_2|x)_{\lambda_j}, \sigma_{k\lambda_k}] \leftarrow \dots \to \beta$$

There is also a path from

$$[\sigma_{1\lambda_1}, \dots, (\tau_1|x|\tau_2|y|\tau_3)_{\lambda_i}, \dots, \sigma_{k\lambda_k}]$$

to $[\sigma_{1\lambda_1}, \dots, (\tau_1|y|\tau_2|x|\tau_3)_{\lambda_i}, \dots, \sigma_{k\lambda_k}]$. Hence for any two vertices v, w in $\mathrm{Sd}\,(C_n(\Gamma_{k,0}))$, we obtain a path from v to w in $\mathrm{Sd}\,(C_n(\Gamma_{k,0}))$ by using these paths. □

Definition 7.4. Let Γ be a finite connected regular cell complex. For a path $P = (e_1, e_2, \dots, e_n)$, the number n is called a *length* of P and is denoted by $\ell(P)$.

Definition 7.5. Let Γ be a finite connected regular cell complex. For two vertices u, v, the *distance* $d_\Gamma(u, v)$ between u and v is defined as follows:

$$d_\Gamma(u, v) = \min\{\ell(P) \mid P \text{ is a path from } u \text{ to } v\}.$$

Lemma 7.6. *Let X be a topological space and A be a subspace of X. Suppose that for any $x \in X$, there exist a point a in A and a path $\ell : I \to X$ such that $\ell(0) = x$ and $\ell(1) = a$. Then X is path connected.*

For a graph Γ containing $\Gamma_{3,0}$, the configuration space $C_n(\Gamma)$ of Γ is path connected.

Theorem 7.7. *Let Γ be a finite connected graph having a vertex v_0 of valency ≥ 3. Then $C_n(\Gamma)$ is path connected.*

Proof. First we prove the case when there is an edge that is not a loop.

If the valency at v_0 is k, there exists an embedding $f : \Gamma_{k,0} \to \Gamma$. For $(x_0, x_1, \dots, x_n) \in C_n(\Gamma)$, we induct on the number α of x_i satisfying $x_i \notin f(\Gamma_{k,0})$.

For $\alpha = 0$, by $(x_0, x_1, \dots, x_n) \in \mathrm{Im}(C_n(f))$, the assumption of Lemma 7.6 is satisfied. We assume that the assumption of Lemma 7.6 is satisfied for $\alpha \leq m - 1$. Consider a point $(x_1, x_2, \dots, x_n)$ in $C_n(\Gamma)$ satisfying $\alpha = m$. Let $x_{i_1}, x_{i_2}, \dots, x_{i_m}$ be the components of $(x_1, x_2, \dots, x_n)$ satisfying $x_{i_j} \notin f(\Gamma_{k,0})$ for all j. Choose a spanning tree T of Γ. It suffices to consider the specialized case where $x_{i_j} \in T$ for some j since if x_{i_j} is not in T for any j, there exists a path from a point x_{i_j} to a vertex in the edge containing x_{i_j} by the connectivity of Γ. Suppose that x_{i_1} satisfies $x_{i_1} \in T$ and the following condition:

$$d_T(x_{i_1}, v_0) = \min\left\{d_T(x_{i_j}, v_0) \;\middle|\; x_{i_j} \in T\right\}.$$

We can choose a path $P = (e_1, e_2, \dots, e_b)$ from v_0 to x_{i_1} in T. Let $x_{h_1}, x_{h_2}, \dots, x_{h_{n-m}}$ be components of $(x_1, x_2, \dots, x_n)$ satisfying $x_{h_a} \in f(\Gamma_{k,0})$.

1. If $x_{h_a} \notin e_1$ for any a, for any $y \in f(\Gamma_{k,0}) \cap e_1$, there exists a path $\ell : I \to \Gamma$ such that $\ell(0) = x_{i_1}$ and $\ell(1) = y$. Hence there exists a path from $(x_1, x_2, \dots, x_{i_1-1}, x_{i_1}, x_{i_1+1}, \dots, x_n)$ to $(x_1, x_2, \dots, x_{i_1-1}, y, x_{i_1+1}, \dots, x_n)$.
2. If $x_{h_a} \in e_1$ for some a, choose x_{h_a} satisfying the following condition:

   $$d_{f(\Gamma_{k,0})\cap e_1}(x_{h_a}, v_0) = \max\left\{d_{f(\Gamma_{k,0})\cap e_1}(x_{h_{a'}}, v_0) \;\middle|\; x_{h_{a'}} \in e_1\right\}$$

   Let $\varphi : D \to \overline{e_1}$ be the cell structure of e_1. Then there exists $t \in D$ such that $\varphi(t) = x_{h_a}$. We choose a positive number $\varepsilon > 0$ such that $\varphi(t+\varepsilon) \in f(\Gamma_{k,0})$. There exists a path $\ell : I \to \Gamma$ such that $\ell(0) = x_{i_1}$ and $\ell(1) = \varphi(t+\varepsilon)$. Hence there exists a path from $(x_1, x_2, \dots, x_n)$ to $(x_1, x_2, \dots, x_{i_1-1}, \varphi(t+\varepsilon), x_{i_1+1}, \dots, x_n)$.

By the inductive hypothesis, the assumption of Lemma 7.6 is satisfied.

Second we prove the case when all edges in Γ are loops, that is $\Gamma = \Gamma_{0,\ell}$.

There is a bijective and continuous map $f : \Gamma_{1,1} \to \Gamma_{0,2}$ from $\Gamma_{1,1}$ to $\Gamma_{0,2}$. f induces a bijective continuous map $C_n(f) : C_n(\Gamma_{1,1}) \to C_n(\Gamma_{0,2})$ from $C_n(\Gamma_{1,1})$ to $C_n(\Gamma_{0,2})$. Hence $C_n(\Gamma_{0,2})$ is path connected.

For $k \geq 3$, there is a bijective and continuous map $g : \Gamma_{k,0} \to \Gamma_{0,k}$ from $\Gamma_{k,0}$ to $\Gamma_{0,k}$. g induces a bijective and continuous map $C_n(g) : C_n(\Gamma_{k,0}) \to C_n(\Gamma_{0,k})$ from $C_n(\Gamma_{k,0})$ to $C_n(\Gamma_{0,k})$. Hence $C_n(\Gamma_{0,k})$ is path connected, and so is $C_n(\Gamma)$. □

8 Fundamental groups of configuration spaces of graphs with a single essential vertex

In this section, we prove Theorem 1.4. The main point is to make explicit the cell complex $\mathrm{Sd}\,(C_n(\Gamma))$ based on the totally normal cell structure on $C_n(\Gamma)$ described in section 6 and then use the equivalence $\mathrm{Sd}\,(C_n(\Gamma)) \simeq C_n(\Gamma)$ obtained from Theorem 4.11 to conclude.

First, we study the dimension of $\mathrm{Sd}\,(C_n(\Gamma_{k,\ell}))$.

Theorem 8.1 ([5]). *Let Γ be a finite connected graph. We have*

$$\dim \mathrm{Sd}\,(C_n(\Gamma)) \leq \min\{n, m\}$$

where m is the number of vertices in Γ. In particular, we have $\dim \mathrm{Sd}\,(C_n(\Gamma_{k,\ell})) = 1$.

In general, for a finite connected 1-dimensional cell complex Γ, the fundamental group of Γ is a free group and the rank of $\pi_1(\Gamma)$ is described by the Euler characteristic of Γ.

Theorem 8.2 ([9]). *Let Γ be a finite connected 1-dimensional cell complex. Then*

1. *The fundamental group $\pi_1(\Gamma)$ of Γ is a free group.*
2. *The rank of the fundamental group $\pi_1(\Gamma)$ is $1 - \chi(\Gamma)$ where $\chi(\Gamma)$ is the Euler characteristic of Γ.*

By Theorem 4.11, Theorem 8.1, and Theorem 8.2, it suffices to calculate the Euler characteristic of $\mathrm{Sd}\,(C_n(\Gamma_{k,\ell}))$ to determine $\pi_1(C_n(\Gamma_{k,\ell}))$.

We define functors to decompose $\mathrm{Sd}\,(C_n(\Gamma_{0,\ell}))$.

Definition 8.3. For $i \in \{1, 2, \dots, n\}$, define a functor

$$\tilde{F}_i : C(C_{n-1}(\Gamma_{1,\ell-1})) \to C(C_n(\Gamma_{0,\ell}))$$

as follows. For $e \in C(C_{n-1}(\Gamma_{1,\ell-1}))_0$, there are

$$e_{\nu_1}, e_{\nu_2}, \dots, e_{\nu_{i-1}}, e_{\nu_{i+1}}, \dots, e_{\nu_n}$$

in $C(\Gamma_{1,\ell-1})_0$, $\sigma \in \Sigma_{\{1,2,\dots,i-1,i+1,\dots,n\}}$, and a partition τ_s of A_s for $1 \le s \le \ell$ such that

$$(e_{\nu_1} \times \dots \times e_{\nu_{i-1}} \times e_{\nu_{i+1}} \times \dots \times e_{\nu_n})\sigma =$$

$$(e^0)^{m_0} \times (e^1_{\lambda_1})^{m_1} \times (e^1_{\mu_1})^{m_2} \times \dots \times (e^1_{\mu_{\ell-1}})^{m_\ell}$$

and

$$e = [\tau_{1\lambda_1}, \tau_{2\mu_1}, \dots, \tau_{\ell\mu_{\ell-1}}]$$

where

$$A_1 = \{x \in \{1, 2, \dots, i-1, i+1, \dots, n\} \mid e_{\nu_x} = e_{\lambda_1}\}$$

and

$$A_s = \{x \in \{1, 2, \dots, i-1, i+1, \dots, n\} \mid e_{\nu_x} = e_{\mu_{s-1}}\}$$

for $2 \le s \le \ell$. Now define $\tilde{F}_i$ by

$$(\tilde{F}_i)_0(e) = [\tau_{2\mu_1}, \dots, \tau_{\ell\mu_{\ell-1}}, (i|\tau_1)_{\mu_\ell}]$$

Proposition 8.4. *Let $F : C(C_n(\Gamma_{0,\ell-1})) \to C(C_n(\Gamma_{0,\ell}))$ be the inclusion functor. Then both BF and $B\tilde{F}_i$ are embeddings.*

Proof. The maps BF and $B\tilde{F}_i$ are injective since F and $\tilde{F}_i$ satisfy the conditions in Proposition 2.12. In general, $\mathrm{Sd}\,(C_n(\Gamma_{k,\ell}))$ is a finite cell complex (i.e. compact) since $C(C_n(\Gamma_{k,\ell}))$ is finite. Hence BF and $B\tilde{F}_i$ are injective continuous maps from a compact space to a Hausdorff space, and embeddings. □

The following two propositions follow immediately by the definition of $\tilde{F}_i$.

Proposition 8.5. *For $1 \le i \le n$,*

$$\mathrm{Im}\, BF \cap \mathrm{Im}\, B\tilde{F}_i = \phi$$

and $\mathrm{Im}\, B\tilde{F}_i \cap \mathrm{Im}\, B\tilde{F}_j = \phi$ if $i \neq j$.

Proposition 8.6. *For functors F and $\tilde{F}_i$, we have*

$$\mathrm{sk}_0(\mathrm{Sd}\,(C_n(\Gamma_{0,l}))) \subset \mathrm{Im}\, BF \amalg \coprod_{i=1}^{n} \mathrm{Im}\, B\tilde{F}_i$$

Definition 8.7. Let Γ be a graph. The set of 1-cells is denoted by $E(\Gamma)$.

The following equation is obtained by Proposition 8.6.

Theorem 8.8. *For $\ell = 2, 3, \dots$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned}\chi(\mathrm{Sd}\,(C_n(\Gamma_{0,\ell}))) &= \chi(\mathrm{Sd}\,(C_n(\Gamma_{0,\ell-1}))) \\ &\quad + n\chi(\mathrm{Sd}\,(C_{n-1}(\Gamma_{1,\ell-1}))) \\ &\quad - n\prod_{i=1}^{n-2}(\ell+i-1)(2\ell+n-3)\end{aligned}$$

Proof. We have the following equation:

$$\begin{aligned}\chi(\mathrm{Sd}\,(C_n(\Gamma_{0,\ell}))) &= \chi(\mathrm{Sd}\,(C_n(\Gamma_{0,\ell-1}))) \\ &\quad + n\chi(\mathrm{Sd}\,(C_n(\Gamma_{1,\ell-1}))) \\ &\quad - \sharp\mathrm{E}(\mathrm{Sd}\,(C_n(\Gamma_{0,\ell})) - X)\end{aligned}$$

where $X = \mathrm{Im}\, BF \amalg \coprod_{i=1}^{n} \mathrm{Im}\, B\tilde{F}_i$. We distinguish two cases for an edge e in $\mathrm{Sd}\,(C_n(\Gamma_{0,\ell})) - X$.

1. The edge e connects $[\tau_{1\mu_1}, \dots, \tau_{\ell-1\mu_{\ell-1}}]$ and $[\tau_{1\mu_1}, \dots, \tau_{\ell-1\mu_{\ell-1}}, (i)_{\mu_\ell}]$. In this case

$$\begin{aligned}A &= 2n\sum_{|a|=n-1}(n-1)! \\ &= 2n\frac{(\ell+n-3)!}{(n-1)!(\ell-2)!}(n-1)! \\ &= 2n\prod_{i=1}^{n-1}(\ell+i-2)\end{aligned}$$

where

$$A = 2n\sum_{|a|=n-1}\frac{(n-1)!}{(n-1-a_1)!} \times \frac{(n-1-a_1)!}{(n-1-a_1-a_2)!} \times \dots \times (a_{\ell-1})!.$$

2. The edge e connects $[\tau_{1\mu_1}, \dots, \tau_{\ell\mu_\ell}]$ and $[\tau_{1\mu_1}, \dots, \tau_{\ell-1\mu_{\ell-1}}, (i|\tau_\ell)_{\mu_\ell}]$. In this case

$$\begin{aligned}&A' \\ &= n\left\{\sum_{|a|=n-1}(n-1)! - \prod_{i=1}^{n-1}(\ell+i-2)\right\} \\ &= n\left\{\frac{(\ell+n-2)!}{(\ell-1)!(n-1)!}(n-1)! - \prod_{i=1}^{n-1}(\ell+i-2)\right\} \\ &= n\left\{\prod_{i=1}^{n-1}(\ell+i-1) - \prod_{i=1}^{n-1}(\ell+i-2)\right\}\end{aligned}$$

where

$$\begin{aligned}A' &= n\sum_{|a|=n-1}\frac{(n-1)!}{(n-1-a_1)!} \times \frac{(n-1-a_1)!}{(n-1-a_1-a_2)!} \\ &\qquad \times \dots \times (a_\ell)! - \prod_{i=1}^{n-1}(\ell+i-2).\end{aligned}$$

Hence the number of edges in $\mathrm{Sd}\,(C_n(\Gamma_{0,l})) - X$ is given by:

$$\begin{aligned}&\sharp\mathrm{E}(\mathrm{Sd}\,(C_n(\Gamma_{0,l})) - X) \\ &= (1) + (2) \\ &= n\left\{\prod_{i=1}^{n-1}(\ell+i-1) + \prod_{i=1}^{n-1}(\ell+i-2)\right\} \\ &= n\prod_{i=1}^{n-2}(\ell+i-1)(\ell+n-2+\ell-1) \\ &= n\prod_{i=1}^{n-2}(\ell+i-1)(2\ell+n-3)\end{aligned}$$

□

We define functors to decompose $\mathrm{Sd}\,(C_n(\Gamma_{k,\ell}))$.

Definition 8.9. For $i \in \{1,2,\ldots,n\}$, define a functor

$$\tilde{G}_i : C(C_{n-1}(\Gamma_{k,\ell})) \to C(C_n(\Gamma_{k,\ell}))$$

as follows: for $e \in C(C_{n-1}(\Gamma_{k,\ell}))_0$, there are $e_{\nu_1}, e_{\nu_2}, \ldots, e_{\nu_{i-1}}, e_{\nu_{i+1}}, \ldots, e_{\nu_n} \in C(\Gamma_{k,\ell})_0$, $\sigma \in \Sigma_{\{1,2,\ldots,i-1,i+1,\ldots,n\}}$, and a partition τ_s of A_s for $1 \le s \le k+\ell$ such that

$$(e_{\nu_1} \times \ldots \times e_{\nu_{i-1}} \times e_{\nu_{i+1}} \times \ldots \times e_{\nu_n})\sigma$$
$$= (e^0)^{m_0} \times (e^1_{\lambda_1})^{m_1} \times (e^1_{\lambda_k})^{m_k} \times (e^1_{\mu_1})^{m_{k+1}} \times \ldots \times (e^1_{\mu_\ell})^{m_{k+\ell}}$$

and

$$e = [\tau_{1\lambda_1}, \ldots, \tau_{k\lambda_k}, \tau_{k+1_{\mu_1}}, \ldots, \tau_{k+\ell_{\mu_\ell}}]$$

where

$$A_s = \{x \in \{1,2,\ldots,i-1,i+1,\ldots n\} \mid e_{\nu_x} = e_{\lambda_s}\}$$

for $1 \le s \le k$ and

$$A_s = \{x \in \{1,2,\ldots,i-1,i+1,\ldots,n\} \mid e_{\nu_x} = e_{\mu_{s-k}}\}$$

for $k+1 \le s \le k+\ell$. Then define $\tilde{G}_i$ by

$$(\tilde{G}_i)_0(e) = [\tau_{1\lambda_1}, \ldots, \tau_{k-1\lambda_{k-1}}, (i|\tau_k)_{\lambda_k}, \tau_{k+1_{\mu_1}}, \ldots, \tau_{k+\ell_{\mu_\ell}}]$$

Proposition 8.10. *Let*

$$G : C(C_n(\Gamma_{k-1,\ell})) \to C(C_n(\Gamma_{k,\ell}))$$

be the inclusion functor. Then both BG and $B\tilde{G}_i$ are embeddings.

Proposition 8.11. *For $1 \le i \le n$,*

$$\mathrm{Im}\,BG \cap \mathrm{Im}\,B\tilde{G}_i = \phi$$

and $\mathrm{Im}\,B\tilde{G}_i \cap \mathrm{Im}\,B\tilde{G}_j = \phi$ if $i \neq j$.

Proposition 8.12. *For functors G and $\tilde{G}_i$, we have*

$$\mathrm{sk}_0(\mathrm{Sd}\,(C_n(\Gamma_{k,\ell}))) \subset \mathrm{Im}\,BG \amalg \coprod_{i=1}^{n} \mathrm{Im}\,B\tilde{G}_i$$

Theorem 8.13. *For $k,n \in \mathbb{N}$, we have*

$$\begin{aligned}\chi(\mathrm{Sd}\,(C_n(\Gamma_{k,\ell}))) &= \chi(\mathrm{Sd}\,(C_n(\Gamma_{k-1,\ell}))) \\ &\quad + n\chi(\mathrm{Sd}\,(C_{n-1}(\Gamma_{k,\ell}))) \\ &\quad - n\prod_{i=1}^{n-1}(\ell+k+i-2)\end{aligned}$$

Proof. We have the following equation:

$$\begin{aligned}\chi(\mathrm{Sd}\,(C_n(\Gamma_{k,\ell}))) &= \chi(\mathrm{Sd}\,(C_n(\Gamma_{k-1,\ell}))) \\ &\quad + n\chi(\mathrm{Sd}\,(C_{n-1}(\Gamma_{k,\ell}))) \\ &\quad - \sharp\mathrm{E}(\mathrm{Sd}\,(C_n(\Gamma_{k,\ell})) - Y)\end{aligned}$$

where $Y = \mathrm{Im}\,BG \amalg \coprod_{i=1}^n \mathrm{Im}\,B\tilde{G}_i$. The edge e connects $[\tau_{1\lambda_1}, \ldots, \tau_{k-1\lambda_{k-1}}, \tau_{k+1_{\mu_1}}, \ldots, \tau_{k+\ell_{\mu_\ell}}]$ and $[\tau_{1\lambda_1}, \ldots, \tau_{k-1\lambda_{k-1}}, (i)_{\lambda_k}, \tau_{k+1_{\mu_1}}, \ldots, \tau_{k+\ell_{\mu_\ell}}]$ for an edge e in $\mathrm{Sd}\,(C_n(\Gamma_{k,\ell})) - Y$. Thus

$$\begin{aligned}&\sharp\mathrm{E}(\mathrm{Sd}\,(C_n(\Gamma_{k,\ell})) - Y) \\ &= n\prod_{|a|=n-1}\frac{(n-1)!}{(n-1-a_1)!} \times \frac{(n-1-a_1)!}{(n-1-a_1-a_2)!} \\ &\qquad \times \ldots \times (a_{\ell+k-1})! \\ &= n\prod_{|a|=n-1}(n-1)! \\ &= n\frac{(\ell+k+n-3)!}{(n-1)!(\ell+k-2)!}(n-1)! \\ &= n\prod_{i=1}^{n-1}(\ell+k+i-2)\end{aligned}$$

as claimed. □

We can now determine $\chi(\mathrm{Sd}\,(C_n(\Gamma_{k,\ell})))$ by Theorem 8.8 and Theorem 8.13.

Theorem 8.14. *For $k = 0,1,2,\ldots$ and $\ell = 1,2,\ldots$, we have*

$$\begin{aligned}&\chi(\mathrm{Sd}\,(C_n(\Gamma_{k,\ell}))) \\ &= -\frac{(\ell+k+n-2)!}{(\ell+k-1)!}((2n-1)(\ell-1)+(n-1)k).\end{aligned}$$

Proof. We induct on n. For $n=1$, $\chi(\mathrm{Sd}\,(C_1(\Gamma_{k,\ell}))) = 1-\ell$ since $\mathrm{Sd}\,(C_1(\Gamma_{k,\ell})) \simeq \Gamma_{k,\ell} \simeq \bigvee_{i=1}^{\ell} S^1$. Assume that

$$\begin{aligned}&\chi(\mathrm{Sd}\,(C_n(\Gamma_{k,\ell}))) = \\ &-\frac{(\ell+k+n-2)!}{(\ell+k-1)!}((2n-1)(\ell-1)+(n-1)k)\end{aligned}$$

for $n \le m-1$. We have

$$\begin{aligned}\chi(\mathrm{Sd}\,(C_m(\Gamma_{0,\ell}))) &= \chi(\mathrm{Sd}\,(C_m(\Gamma_{0,\ell-1}))) \\ &\quad - m\prod_{i=1}^{m-2}(\ell+i-1)(\ell-1)(2m-1)\end{aligned}$$

since

$$\begin{aligned}\chi(\mathrm{Sd}\,(C_m(\Gamma_{0,\ell}))) &= \chi(\mathrm{Sd}\,(C_m(\Gamma_{0,\ell-1}))) \\ &\quad + m\chi(\mathrm{Sd}\,(C_{m-1}(\Gamma_{1,\ell-1}))) \\ &\quad - m\prod_{i=1}^{m-2}(\ell+i-1)(2\ell+m-3)\end{aligned}$$

and

$$\begin{aligned}&\chi(\mathrm{Sd}\,(C_{m-1}(\Gamma_{1,\ell-1}))) = \\ &-\frac{(\ell+m-3)!}{(\ell-1)!}((2m-3)(\ell-2)+(m-2))\end{aligned}$$

by the inductive hypothesis. Then

$$\chi(\mathrm{Sd}\,(C_m(\Gamma_{0,\ell}))) = -\frac{(\ell+m-2)!}{(\ell-1)!}(2m-1)(\ell-1)$$

since the sequence $\{\chi(\mathrm{Sd}\,(C_m(\Gamma_{0,\ell})))\}_{\ell=1}^{\infty}$ is satisfying the following recurrence relation $\chi(\mathrm{Sd}\,(C_m(\Gamma_{0,1}))) = 0$ and

$$\chi(\mathrm{Sd}\,(C_m(\Gamma_{0,\ell}))) = \chi(\mathrm{Sd}\,(C_m(\Gamma_{0,\ell-1}))) \\ -m\prod_{i=1}^{n-2}(\ell+i-1)(\ell-1)(2m-1).$$

For $k \neq 0$,

$$\chi(\mathrm{Sd}\,(C_m(\Gamma_{k,\ell}))) = \chi(\mathrm{Sd}\,(C_m(\Gamma_{k-1,\ell}))) \\ - m\prod_{i=2}^{m-1}(\ell+k+i-2)(2\ell+k-2)(m-1)$$

since

$$\chi(\mathrm{Sd}\,(C_m(\Gamma_{k,\ell}))) = \chi(\mathrm{Sd}\,(C_m(\Gamma_{k-1,\ell}))) \\ + m\chi(\mathrm{Sd}\,(C_{m-1}(\Gamma_{k,\ell}))) - m\prod_{i=1}^{m-1}(\ell+k+i-2)$$

and

$$\chi(\mathrm{Sd}\,(C_{m-1}(\Gamma_{k,\ell}))) = \\ -\frac{(\ell+k+m-3)!}{(\ell+k-1)!}((2m-3)(\ell-1)+(m-2)k)$$

by the inductive hypothesis. Then

$$\chi(\mathrm{Sd}\,(C_m(\Gamma_{k,\ell}))) = \\ -\frac{(\ell+k+m-2)!}{(\ell+k-1)!}((2m-1)(\ell-1)+(m-1)k)$$

since the sequence $\{\chi(\mathrm{Sd}\,(C_m(\Gamma_{k,\ell})))\}_{k=0}^{\infty}$ is satisfying the following recurrence relation

$$\chi(\mathrm{Sd}\,(C_m(\Gamma_{0,\ell}))) = -\frac{(\ell+m-2)!}{(\ell-1)!}(2m-1)(\ell-1), \quad \text{and}$$

$$\chi(\mathrm{Sd}\,(C_m(\Gamma_{k,\ell}))) = \chi(\mathrm{Sd}\,(C_m(\Gamma_{k-1,\ell}))) \\ -m\prod_{i=2}^{m-1}(\ell+k+i-2)(2\ell+k-2)(m-1).$$

□

As a consequence, we determine $\pi_1(C_n(\Gamma_{k,\ell}))$.

Theorem 8.15. *If $(k,\ell) \neq (1,0),(2,0),(0,1)$, then*

$$\pi_1(C_n(\Gamma_{k,\ell})) \cong F_m$$

where

$$m = 1 + \frac{(\ell+k+n-2)!}{(\ell+k-1)!}((2n-1)(\ell-1)+(n-1)k).$$

Next, we study the projection map $p : C_n(\Gamma_{k,\ell}) \to C_n(\Gamma_{k,\ell})/\Sigma_n$ to determine the fundamental group $\pi_1(C_n(\Gamma_{k,\ell})/\Sigma_n)$ of $C_n(\Gamma_{k,\ell})/\Sigma_n$.

Since Σ_n acts freely on the path-connected and Hausdorff space $C_n(X)$, it follows that the projection $p : C_n(X) \to C_n(X)/\Sigma_n$ is a covering space projection. By the classification theorem of covering spaces [[9], section 1.3], we have

$$[\pi_1(C_n(X)/\Sigma_n, p(*)), p_*(\pi_1(C_n(X),*))] \\ = \sharp p^{-1}(p(*)) = |\Sigma_n| = n!$$

The following result is used in order to determine the rank of $\pi_1(C_n(\Gamma_{k,\ell})/\Sigma_n)$.

Proposition 8.16 ([11])**.** *Let H be a subgroup of a free group F_k. If $[F_k : H] = n < \infty$, then*

1. *H is also free group.*
2. *$H = F_m$ where $m = 1 - n + nk$.*

Theorem 8.17. *For $(k,\ell) \neq (1,0),(2,0),(0,1)$ and $n \geq 3$,*

$$\pi_1(C_n(\Gamma_{k,\ell})/\Sigma_n) \cong F_m$$

where

$$m = 1 + \frac{1}{n!}\frac{(\ell+k+n-2)!}{(\ell+k-1)!}((2n-1)(\ell-1)+(n-1)k)$$

Proof. We have an isomorphism

$$\pi_1(C_n(\Gamma_{k,\ell})) \cong p_*(\pi_1(C_n(\Gamma_{k,\ell})))$$

since the projection map $p : C_n(\Gamma_{k,\ell}) \to C_n(\Gamma_{k,\ell})/\Sigma_n$ is a covering space. Assume that m is the rank of $\pi_1(C_n(\Gamma_{k,\ell})/\Sigma_n)$. Then

$$[\pi_1(C_n(\Gamma_{k,\ell})/\Sigma_n) : p_*(\pi_1(C_n(\Gamma_{k,\ell})))] = [F_m : F_{m'}] \\ = n!$$

where

$$m' = 1 + \frac{(\ell+k+n-2)!}{(\ell+k-1)!}((2n-1)(\ell-1)+(n-1)k).$$

Hence

$$m = 1 + \frac{1}{n!}\frac{(\ell+k+n-2)!}{(\ell+k-1)!}((2n-1)(\ell-1)+(n-1)k)$$

since $m' = 1 - n! + n!m$. □

References

[1] Aaron Abrams. *Configuration Spaces and Braid Groups of Graphs*, PhD thesis, University of California at Berkeley, 2000.

[2] Aaron Abrams and Robert Ghrist. *Finding topology in factory: configuration space*, Amer. Math. Monthly, volume 109, pages 140–150.

[3] A Arhangel'skii. *Some types of factor mappings and the relations between classes of topological spaces*, Dokl. Akad. Nauk SSSR, 153 (1963), 743–746. [Arh63]

[4] John Crisp and Bert Wiest. *Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups*, Algebraic and Geometric Topology, volume 4 (2004), 439–472

[5] Mizuki Furuse, Takashi Mukouyama, and Dai Tamaki. *Totally Normal Cellular Stratified Spaces and Applications to the Configuration Space of Graphs*, arXiv:1312.7368v2.

[6] E. Fadell and L. Neuwirth. *Configuration spaces. Math. Scand*, (10) (1962), 111–118.

[7] Daniel Farley and Lucas Sabalka. On the cohomology rings of tree braid groups, J.Pure Appl. Algebra **212** (2007), (1): 53–71.

[8] Robert Ghrist. *Configuration spaces and braid groups on graphs in robotics*, in Knots, braids, and mapping class groups-papers dedicated Joan S. Birman (New York, 1998), volume 24 of AMS/IP Stud. Adv. Math., pages 29-40, (2001).

[9] Allen Hatcher. Algebraic Topology. Cambridge University Press, Cambridge, 2002.

[10] J. H. Kim, K. H. Ko, and H. W. Park. *Graph braid groups and right-angled Artin groups*, Trans. Amer. Math. Soc., 364(1) (2012), 309–360.

[11] J. P. May. A concise course in algebraic topology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[12] Ernest Michael. *Bi-quotient maps and Cartesian products of quotient maps*, Ann. Inst. Fourier (Grenoble), 18(fasc. 2):287–302 vii (1968).

[13] J. Milnor. *The geometric realization of a semi-simplicial complex*, Ann. of Math. (2) **65** (1957), 356–362.

[14] Takashi Mukouyama. *About configuration spaces of graphs*, Master's thesis, Shinshu University, 2013.

[15] Dai Tamaki. *Cellular Stratified Spaces I: Face Categories and Classifying Spaces*, arXiv:1106.3772v3.

[16] Dai Tamaki. *Cellular Stratified Spaces II: Basic Constructions*, arXiv:1111.4774v3.

T. Unohara
Department of Mathematics
Shinshu University,
Matsumoto, 390-8621
Japan
E-mail address: unohara_7283@yahoo.co.jp