Graduate J. Math. **1** (2016), 54 – 58

The Space of Closed Subgroups of $\mathbb{R}^2$ and Seifert Invariants

Benoît Kloeckner

Abstract

In this note we give a version of the proof of the Hubbard-Pourezza Theorem, introduced below, using Seifert fibration. Our main goal is to compute explicitly and directly Seifert invariants in this nice example. The fact that it is fairly hard to find such a direct computation in the literature is our main motivation.

MSC 2010. Primary 57S05; Secondary 32S50.

1 Introduction

Let G be a topological group whose neutral element is denoted by 0 (although G need not be abelian). Its *Chabauty space* $\mathscr{C}(G)$ is the set of closed subgroups of G endowed with the following topology: the neighborhoods of a point $\Gamma \in \mathscr{C}(G)$ are the sets

$$\mathcal{N}_U^K(\Gamma) = \{\Gamma' \in G \mid \Gamma' \cap K \subset \Gamma \cdot U \text{ and } \Gamma \cap K \subset \Gamma' \cdot U\}$$

where K runs over the compact subsets of G and U runs over the neighborhoods of 0. In words, Γ' is very close to Γ if, on a large compact set, every of its elements is in a uniformly small neighborhood of an element of Γ, and conversely. The preprint [13] contains a more detailed account of this topology.

The simplest example of a Chabauty space is that of the line: $\mathscr{C}(\mathbb{R})$ contains the trivial subgroup $\{0\}$, the discrete groups $\alpha\mathbb{Z}$ and the total group $\mathbb{R}$. Two discrete groups $\alpha\mathbb{Z}$ and $\beta\mathbb{Z}$ are close one to another when α and β are close, a neighborhood of $\{0\}$ consists in the set of $\alpha\mathbb{Z}$ with large α (and we define $\infty\mathbb{Z} = \{0\}$) and a neighborhood of $\mathbb{R}$ consists in the set of $\alpha\mathbb{Z}$ with small α (and we define $0\mathbb{Z} = \mathbb{R}$). Putting all this together, we see that $\mathscr{C}(\mathbb{R})$ is homeomorphic to a closed interval.

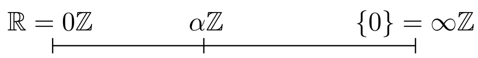

Fig. 1: Chabauty space of $\mathbb{R}$.

Only for a few groups G do we have a precise description of $\mathscr{C}(G)$. Works of Bridson, de la Harpe and Kleptsyn [3] and Haettel [11] have added to the list the three-dimensional Heisenberg group and $\mathbb{R} \times \mathbb{Z}$ respectively, but the topology of $\mathscr{C}(\mathbb{R}^n)$ is unknown for $n > 2$ (although we proved in [17] that it is simply connected). Even $\mathscr{C}(\mathbb{R}^2)$ is not easy to describe; it was tackled by Hubbard and Pourezza [15] who proved the following.

Theorem 1.1 (Hubbard-Pourezza)**.** *Let $\mathscr{C}$ be the Chabauty space of $\mathbb{R}^2$ and $\mathscr{L}$ be the subset of lattices. The topological pair $(\mathscr{C}, \mathscr{C} \smallsetminus \mathscr{L})$ is homeomorphic to the suspension of (S^3, K) where K is a trefoil knot in the 3-sphere. In particular, $\mathscr{C}$ is a 4-sphere.*

Let us recall some definitions. A *topological pair* is a pair (X, Y) of topological spaces where Y is a subset of X (endowed with the induced topology). Two topological pairs (X, Y) and (X', Y') are *homeomorphic* if there is a homeomorphism $\Phi : X \to X'$ that maps Y onto Y'. The *(open) cone* over X is the quotient cX of $X \times [0, 1)$ by the relation $(x_0, 0) \sim (x_1, 0)$, while the *suspension* of X is the quotient sX of $X \times [0, 1]$ by the relations $(x_0, 0) \sim (x_1, 0)$ and $(x_0, 1) \sim (x_1, 1)$ for all $x_0, x_1 \in X$. If Y is a subset of X, then sY embeds naturally in sX and the resulting topological pair (sX, sY) is called the suspension of (X, Y). The Hubbard-Pourezza theorem shows in particular that the set of non-lattices is a 2-sphere that is *non-tamely* embedded in $\mathscr{C} \simeq S^4$.

The goal of this note is to give a proof of this theorem using Seifert fibration. This proof is not really original, it is even alluded to in the paper of Hubbard and Pourezza. The topology of the subspace of lattices is a very classical topic, see for example [23], [13] which also contains a detailed version of the original proof of Hubbard-Pourezza's result, or [22] which also links to Seifert fibrations. However we could not find the explicit computation of Seifert invariants, as presented here, in the literature.

2 The Chabauty space of $\mathbb{R}^2$ is a 4-sphere

2.1 Definitions and notations

In this section, we denote by $\mathscr{C}$ the Chabauty space of $\mathbb{R}^2$. A closed subgroup of $\mathbb{R}^2$ is of one of the following types:

- $(0,0)$: the trivial subgroup 0 ;
- $(0,1)$: isomorphic to $\mathbb{Z}$;
- $(0,2)$: isomorphic to $\mathbb{Z}^2$ (these are the lattices) ;
- $(1,0)$: isomorphic to $\mathbb{R}$;
- $(1,1)$: isomorphic to $\mathbb{R}\times\mathbb{Z}$;
- $(2,0)$: the total group $\mathbb{R}^2$.

Each type is an orbit of the action of $\mathrm{GL}(2;\mathbb{R})$ on $\mathscr{C}$. The set of lattices is $\mathscr{L} =: \mathscr{C}^{(0,2)}$, its complement is denoted by $\mathscr{H}$.

A closed subgroup Γ of $\mathbb{R}^2$ has a determinant, or covolume, covol(Γ). If Γ is a lattice, it is its usual determinant, that is the determinant of any direct base of Γ. It is 0 if Γ is isomorphic to $\mathbb{R}\times\mathbb{Z}$ or $\mathbb{R}^2$, and ∞ if Γ is isomorphic to $\mathbb{Z}$ or 0. By convention, covol(Γ) takes simultaneously all values in $[0,\infty]$ if Γ is isomorphic to $\mathbb{R}$. So defined, the levels of covol are closed in $\mathscr{C}$. Outside the set $\mathscr{R} := \mathscr{C}^{(1,0)}$ of subgroups isomorphic to $\mathbb{R}$, covol is a continuous function.

Let $\mathscr{C}_{\geqslant 1}$, respectively $\mathscr{C}_{\leqslant 1}$, be the subsets of $\mathscr{C}$ defined by covol $\geqslant 1$ and covol $\leqslant 1$. These sets both contain $\mathscr{R}$. Let $\mathscr{H}_{\geqslant 1} = \mathscr{H}\cap\mathscr{C}_{\geqslant 1}$ be the set of subgroups isomorphic to $\mathbb{R}$, $\mathbb{Z}$ or 0, and $\mathscr{H}_{\leqslant 1} = \mathscr{H}\cap C_{\leqslant 1}$ be the set of subgroups isomorphic to $\mathbb{R}$, $\mathbb{R}\times\mathbb{Z}$ or $\mathbb{R}^2$.

Let $\mathscr{L}_1$ be the set of covolume 1 lattices, and $\mathscr{C}_1$ its closure. Then $\mathscr{C}_1$ is the union of $\mathscr{L}_1$ and of the set $\mathscr{R}$.

We use the usual identification $\mathbb{R}^2\simeq\mathbb{C}$, so that any subgroup isomorphic to $\mathbb{R}$ can be written in the form $e^{i\theta}\mathbb{R}$.

We also define the norm (or systol)

$$N(\Gamma) = N_1(\Gamma) = \inf\{|x| \mid x\in\Gamma\setminus\{0\}\}$$

It is a continuous functions taking its values in $[0,\infty]$. Let $\mathscr{C}^1$ be the set of norm 1 subgroups of $\mathbb{R}^2$. A point of $\mathscr{C}^1$ is either isomorphic to $\mathbb{Z}$, or a lattice. We denote by $\mathscr{Z}^1$ the set $\mathscr{C}^1\setminus\mathscr{L}$.

Figure 2 sums up this notations.

The proof of Theorem 1.1 is in two parts. We first prove that the topological pair $(\mathscr{C},\mathscr{H})$ is the suspension of $(\mathscr{C}^1,\mathscr{Z}^1)$, then that the latter is homeomorphic to (S^3,K) where K is a trefoil knot.

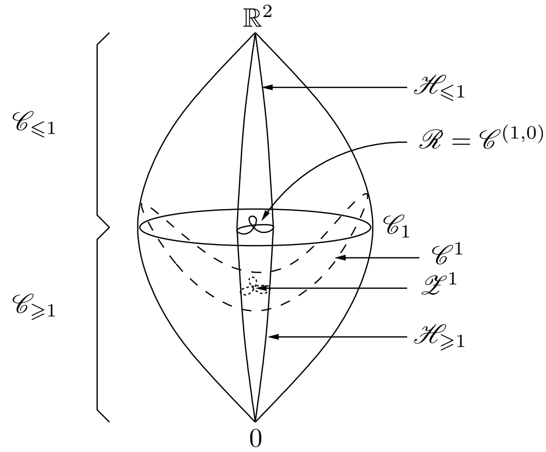

Fig. 2: Sum up of notations

2.2 The Chabauty space of $\mathbb{R}^2$ is a suspension

In this first part of the proof, which is not our main motivation, the proof will be given only few details.

Lemma 2.1. *The topological pair $(\mathscr{C}_{\geqslant 1},\mathscr{H}_{\geqslant 1})$ is homeomorphic to the cone over $(\mathscr{C}_1,\mathscr{R})$.*

Proof. We consider the map

$$\begin{array}{rcl}\Phi:\mathscr{C}_1\times[0,\infty] & \to & \mathscr{C}_{\geqslant 1}\\ (\Gamma_1,t) & \mapsto & \begin{cases}\left(\frac{t}{N(\Gamma_1)}+1\right)\Gamma_1 \text{ if } \Gamma_1\in L_1\\ te^{i\theta}\mathbb{Z} \text{ if } \Gamma_1 = e^{i\theta}\mathbb{R}\end{cases}\end{array}$$

where by convention $0e^{i\theta}\mathbb{Z} = e^{i\theta}\mathbb{R}$ and $\infty\Gamma = 0$ if Γ is discrete.

This map is continuous, maps $\mathscr{C}_1\times\{0\}$ onto $\mathscr{C}_1$ and $\mathscr{R}\times[0,\infty]$ onto $\mathscr{H}_{\geqslant 1}$. It induces a continuous bijection $\tilde{\Phi}$ from the quotient of $\mathscr{C}_1\times[0,\infty]$ by the relation $(\Gamma_1,\infty)\sim(\Gamma_1',\infty)$ onto $\mathscr{C}_{\geqslant 1}$. Since the latter is compact, $\tilde{\Phi}$ is a homeomorphism between the cone over $(\mathscr{C}_1,\mathscr{R})$ and $(\mathscr{C}_{\geqslant 1},\mathscr{H}_{\geqslant 1})$. □

Lemma 2.2. *The topological pair $(\mathscr{C}_1,\mathscr{R})$ is homeomorphic to $(\mathscr{C}^1,\mathscr{Z}^1)$.*

Proof. The map $\Psi:\mathscr{C}^1\to\mathscr{C}_1$ that assigns to Γ the only $t\Gamma$ of unit covolume ($t=0$ if Γ is isomorphic to $\mathbb{Z}$, $t = \mathrm{covol}(\Gamma)^{-1/2}$ otherwise) is continuous and a bijection. By compacity of $\mathscr{C}_1$, closed in $\mathscr{C}$, it is a homeomorphism. □

Proposition 2.3. *The topological pair $(\mathscr{C},\mathscr{H})$ is homeomorphic to the suspension of $(\mathscr{C}^1,\mathscr{Z}^1)$.*

Proof. We can either reproduce the previous arguments to prove that $(\mathscr{C}_{\leqslant 1},\mathscr{H}_{\leqslant 1})$ is also a cone over $(\mathscr{C}^1,\mathscr{Z}^1)$ or use the duality $*$ which maps $\mathscr{C}_{\geqslant 1}$ on $\mathscr{C}_{\leqslant 1}$ and preserves $\mathscr{L}$. □

2.3 Subgroups of unit norm

To get Theorem 1.1, we have left to prove the following.

Proposition 2.4. *The topological pair* $(\mathscr{C}^1, \mathscr{Z}^1)$ *is homeomorphic to* (S^3, K).

The proof runs over the rest of the Section. We shall describe $\mathscr{C}^1$ as a Seifert fibration (see for example [1] for an introduction to Seifert fibrations). Let Γ be a point of $\mathscr{C}^1$. The isometry group SO(2) acts on $\mathscr{C}^1$, and up to a rotation we can assume that $1 \in \Gamma \subset \mathbb{C}$. Then Γ is determined by the choice of a second vector in the fundamental domain

$$D = \{z \in \mathbb{C}; |z| \geqslant 1 \text{ and } -1/2 \geqslant \mathrm{Re}(z) \geqslant 1/2\} \cup \{\infty\}$$

where $z = \infty$ means that Γ is isomorphic to $\mathbb{Z}$ (figure 3). Identifying the points of D that represent the same Γ leads to the quotient of D by the relation $z \sim z - 1$ if $\mathrm{Re}(z) = 1/2$ and $z \sim -\bar{z}$ if $|z| = 1$, turning it into a 2-sphere denoted by B, that will be the base of the Seifert fibration.

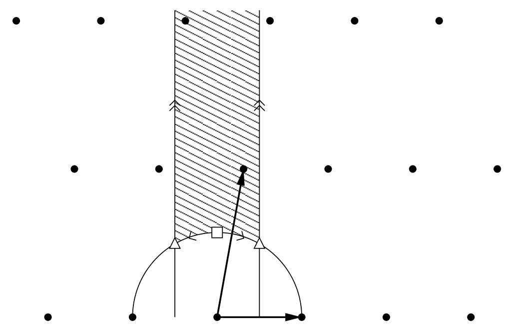

Fig. 3: Fundamental domain: the vertical lines and the circle arcs are glued according to the arrows, □ and △ are the singular points.

The kernel of the action of SO(2) is reduced to $\{\pm 1\}$, and the quotient gives an action of the circle that is almost free: the only points of $\mathscr{C}^1$ that have nontrivial stabilizers are the triangular lattices (stabilizer of order 3) and the square lattices (stabilizer of order 2). It follows that $\mathscr{C}^1$ is a Seifert fibration with base $B \simeq S^2$ and two singular fibers of order 2 and 3, and where $\mathscr{Z}^1$ is a regular fiber. The unnormalized Seifert invariants of $\mathscr{C}^1$ are $(0|(2, \beta_1); (3, \beta_2))$ and we have left to find the rational Euler number $\beta_1/2 + \beta_2/3$ to determine $(\mathscr{C}^1, \mathscr{Z}^1)$.

We first choose a cross-section of the regular part of the Seifert fibration. It would be natural to lift each point u in the fundamental domain to the subgroup generated by u and 1, but this would not define a continuous cross-section. The gluing of the unit circle indeed identifies, for all $\theta \in [0, \pi/6]$, the subgroups $1\mathbb{Z} + e^{i(\pi/2-\theta)}\mathbb{Z}$ and $1\mathbb{Z} + e^{i(\pi/2+\theta)}\mathbb{Z}$ by a rotation of angle $\pi/2+\theta$. We shall therefore modify this cross-section in a neighborhood of one of the circular arcs of D.

Let $S^1 = \mathbb{R}/\pi\mathbb{Z}$ be the quotient $\mathrm{SO}(2)/\{\pm 1\}$, D' be the fundamental domain D minus the singular points (i, $e^{i\pi/3}$ and $e^{2i\pi/3}$) and B' be the base B minus the two singular points (corresponding to i and $e^{i\pi/3} \sim e^{2i\pi/3}$). We choose a continuous map $f : D' \to [0, \pi/2]$ that is constant with value 0 except in a neighborhood of the arc $\left\{e^{i(\pi/2+\theta)} \;\middle|\; \theta \in]0, \pi/6[\right\}$, where it satisfies $f(e^{i(\pi/2+\theta)}) = \pi/2 - \theta$. We then define a cross-section $\sigma : B' \to \mathscr{C}^1$ by $\sigma(u) = e^{if(u)}(1\mathbb{Z}+u\mathbb{Z})$. It is continuous since

$$\begin{aligned} \sigma(e^{i(\pi/2+\theta)}) &= e^{i(\pi/2-\theta)}(1\mathbb{Z} + e^{i(\pi/2+\theta)}\mathbb{Z}) \\ &= e^{i(\pi/2-\theta)}\mathbb{Z} + 1\mathbb{Z} \\ &= \sigma(e^{i(\pi/2-\theta)}). \end{aligned}$$

Let b be the homotopy class in $\mathscr{C}^1$ of a regular fiber, d_1 and d_2 be the homotopy classes defined by σ on the boundary of $\mathscr{C}^1_\bullet = \mathscr{C}^1 \setminus \{F_1, F_2\}$ where F_1 and F_2 are invariant neighborhoods of the singular fibers of order 2 and 3, respectively.

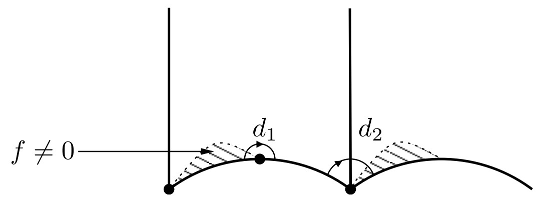

Fig. 4: The cross-section σ defines homotopy classes in the boundary of $\mathscr{C}^1_\bullet$.

In ∂F_1 and ∂F_2 respectively, we get that $2d_1 + b$ and $3d_2 - b$ are homotopic to meridians (see figure 5 where F_1 and F_2 are pictured with coordinates $(u, \varphi) \in B \times \mathbb{R}/\pi\mathbb{Z} \mapsto e^{i\varphi}(1\mathbb{Z} + u\mathbb{Z})$, with the suitable identifications). It follows that $\mathscr{C}^1$ has unnormalized Seifert invariants $(0|(2, 1), (3, -1))$ and rational Euler number equal to $1/2 - 1/3 = 1/6$.

We shall know exhibit a very classical Seifert fibration on S^3 whose regular fibers are trefoil knots, that has base S^2, two singular fibers of order 2 and 3 and rational Euler number 1/6. Since a Seifert fibration is determined by these data, we will conclude that $(\mathscr{C}^1, \mathscr{Z}^1)$

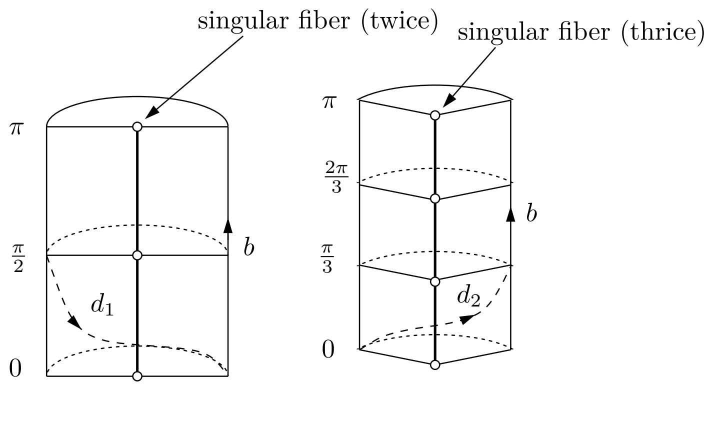

Fig. 5: Neighborhood F_1 and F_2 of the singular fibers

is homeomorphic to (S^3, K).

Consider the following action of the circle $\mathbb{R}/\mathbb{Z}$ on S^3, identified to the unit sphere of $\mathbb{C}^2$:

$$s \cdot (z_1, z_2) = (e^{2\pi m_1 is} z_1, e^{2\pi m_2 is} z_2)$$

with $m_1 = 2$ and $m_2 = 3$. The stabilizer of almost every point is trivial, the exceptions being the polar orbits $(z_1, 0)$ and $(0, z_2)$. If m_1 and m_2 where equal to 1, we would get the Hopf fibration where the non-polar orbits are Villarceau circles of the tori $|z_1/z_2| = c$, where c runs over $[0, \infty]$. Taking $m_1 = 2$ and $m_2 = 3$, we replaced the Villarceau circle by toric knots, here trefoil knots (figure 6).

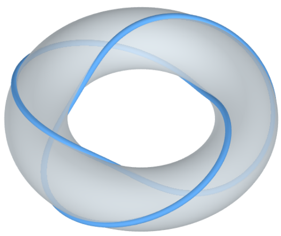

Fig. 6: The torus knot $(2, 3)$ is a trefoil knot.

We see that the regular part of the base is foliated by the circles obtained by quotienting the tori $|z_1/z_2| = c$ by the action of S^1, and is therefore an annulus. One can see this annulus as the S^2 base of the Hopf fibration minus two points for the singular fibers.

Let us compute the Seifert invariants of this action, which are surprisingly difficult to find in the litterature. We use a representation found in [18].

Let $T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ be the standard 2-torus equiped with the foliation by straight lines of slope $3/2$. If we denote by x the homotopy class of

$$\begin{array}{rcl} \mathbb{R}/\mathbb{Z} & \to & T^2 \\ t & \mapsto & (t, 0) \end{array}$$

and by y the homotopy class of

$$\begin{array}{rcl} \mathbb{R}/\mathbb{Z} & \to & T^2 \\ t & \mapsto & (0, t) \end{array}$$

the homotopy class of any leave of this foliation is $\ell = 2x + 3y$.

In the space $T^2 \times [0, 1]$ define $T_t := \mathbb{T}^2 \times \{t\}$, endowed with the above foliation for $t \in (0, 1)$. Let $\Pi : T^2 \times [0, 1] \to S^3$ be the mapping defined as follows. First, Π contracts T_0 to the singular fiber $\{(0, z_2) \mid |z_2| = 1\}$ and T_1 to the singular fiber $\{(z_1, 0) \mid |z_1| = 1\}$ with $\Pi(a, b, 0) = (0, e^{2i\pi b})$ and $\Pi(a, b, 1) = (e^{2i\pi a}, 0)$. Second, it maps T_t to a torus defined by $|z_1/z_2| = c(t)$ with c an increasing continuous function such that $c(t) \to 0$ (resp. $+\infty$) when $t \to 0$ (resp. 1), and maps the foliation of T_t to the Seifert foliation in S^3. Think of $T^2 \times [0, 1]$ as a blow-up of S^3 along the singular fibers.

The point is that in this presentation, one can give explicitely a cross-section of the Seifert fibration over the regular part: just consider the set

$$\{(s, 2s, t) \mid s \in \mathbb{R}/\mathbb{Z}, t \in (0, 1)\} \subset T^2 \times (0, 1)$$

This set intersects each of the T_t along a straight line homotopic to $x + 2y$, which intersects each $2x + 3y$ line once, thus it does define a section.

In the boundary of a neighborhood of T_0, the section defines a curve homotopic to $d_0 = -x - 2y$ (the sign depends upon the choice of orientation). Since $\ell = 2x + 3y$ is the homotopy class of a regular fiber, we have $3d_0 + 2\ell = x$, a meridian. Similarly, in the boundary of a neighborhood of T_1, the section defines a curve homotopic to $d_1 = x + 2y$ and $2d_1 - \ell = y$ is a meridian.

Therefore, this Seifert fibration has unnormalized invariants $(0|(3, 2), (2, -1))$ and rational Euler number $2/3 - 1/2 = 1/6$ as needed.

Remark 2.5. As we said in the introduction, it is well known that the homogeneous space $\mathrm{SL}(2;\mathbb{R})/\mathrm{SL}(2;\mathbb{Z})$ is homeomorphic to the complement of a trefoil knot in S^3. Here the difficulty is to prove that when gluing the fiber $\mathscr{L}^1$ we do get a sphere and not some other 3-manifold obtained by surgery along a trefoil knot.

Remark 2.6. Christopher Tuffley studied [28] the spaces $\exp_k(S^1)$ of all non-empty subset of the circle of cardinality at most k. In particular, he proved using Seifert fibrations that $\exp_3(S^1)$ is a 3-sphere, its subset $\exp_1(S^1)$ being a trefoil knot.

The similarity with Proposition 2.4 is not fortuitous: Jacob Mostovoy proved [21] by a simple geometric argument that $(\exp_3(S^1), \exp_1(S^1))$ is homeomorphic to $(\mathscr{C}^1, \mathscr{L}^1)$. Combining these two results one gets another Seifert fibration proof of Proposition 2.4. Note that even the Seifert part is somewhat different from ours, since it is first proved that $\exp_3(S^1)$ is simply connected, which reduces drastically its possible Euler numbers.

Remark 2.7. A nice feature of the study of $\exp_3(S^1)$ is that its subset $\exp_2(S^1)$ is easily seen to be a Möbius strip, with boundary $\exp_1(S^1)$: we recover the fact that a trefoil knot bounds a Möbius strip. This can be seen in $(\mathscr{C}^1, \mathscr{L}^1)$ as well: over the vertical line $L = \{iy \mid y \in [1, +\infty]\}$ of the base B, the Seifert fibration is a closed Möbius strip whith boundary $\mathscr{L}^1$, obtained by identifying antipodal points of the $(y = 1)$ boundary component of the strip $L \times S^1$.

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Benoît Kloeckner

Université Paris-Est,
Laboratoire d'Analyse et de Mathématiques Appliquées,
(UMR 8050), UPEM, UPEC, CNRS, F-94010,
Créteil,France

E-mail address: `benoit.kloeckner@u-pec.fr`