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A short review on boundary behavior of linear diffusion processes

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Abstract

In this paper, we present in a complete and synthetic way the modern classification of boundaries of one dimensional diffusion processes. This classification is proved using the scale function as well as the speed measure associated to a one dimensional diffusion process. In order to highlight the behavior of a diffusion process, we follow a numerical approach with graphical illustrations. To this end, different schemes are used to approximate the paths of a diffusion process such as the Euler-Maruyama and Milstein algorithms which are used to approximate the Wright-Fisher diffusion process near "exit", "regular", "entrance" or "natural" boundaries (definitions in the text). Finally, we offer well-organized tables on the nature of any boundary (closed or open) of a given sub-interval of the state space, with appropriate conditions. We believe that these contributions are very useful to better understand the physical and natural meaning of boundary classification.

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Introduction

Diffusion processes have numerous applications ranging from physics and biology to the economy and social sciences. They are used for approximating or modeling phenomena assumed to evolve randomly and continuously in time. During the last thirty years or so diffusion models have been applied intensively in :

- Mathematical finance for describing stock prices, interest rates, etc.
- Mathematical Biology to approximate reproduction, genetic disease, etc. in several populations.

In this paper, we are interested in the study of the boundary classification of the linear diffusion process $(X_t)_{t\geq 0}$ of which dynamics are described by:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t, X_0 = x \in (x_l, x_r) \quad (0.1)$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion, $\mu(.)$ and $\sigma(.)$ are respectively the drift and the diffusion coefficient functions. We shall assume that the state space $\mathcal{I}$ of $(X_t)_{t\geq 0}$ is a finite or infinite interval with endpoints $-\infty \leq a < b \leq +\infty$ and assume also that $\mathbb{I} = (x_l, x_r)$ is any subinterval of the state space $\mathcal{I}$. Besides, we assume that $(X_t)_{t\geq 0}$ is adapted to a filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions. Let us introduce:

$$X_a := \limsup_{t\to a} X_t \quad \text{and} \quad X_b := \limsup_{t\to b} X_t.$$

We will address the relevance of Feller's work on boundary classification of one-dimensional diffusion processes following the stochastic differential equation (0.1), within a general context of mathematics and biology. Feller begins his work on boundary behavior of one-dimensional diffusions in the paper [8] by studying a particular forward equation. He shows that the nature of boundary forces the stochastic differential equation to change its character completely for some proposed condition in such a way that the boundary 0 is exit, regular or entrance. In the case where the diffusion process takes non-negative values Feller develops [7] motivated by application in biology. [9] provides a review of Feller's boundaries classification results for Brownian motion in $]0, \infty[$. Sixteen years later, [13] propose a table which presents some comparison between the American and Russian boundaries classifications, and they illustrate this table by some models in biology and finance by studying the appropriate boundaries condition.

From the theoretical point of view, same assumptions on the structure of the diffusion coefficient or the drift serve as a physical interpretation for the behavior of diffusive trajectories near the boundaries. The purpose of the present paper is to give a complete classification of boundary types and boundary behavior of one-dimensional diffusion processes. Furthermore we illustrate graphically all types of boundaries (e.g Brownian motion, Wright-Fisher diffusion process and geometric Brownian motion) by providing an example, and we will give all possible state space boundary form with appropriate conditions.

The outline of the paper is as follows:

In section 1 we give an overview on diffusion processes. We introduce in subsection 1.1 some notions on diffusion process, present a solution of the SDE (0.1) and make some hypothesis to ensure the existence of a unique solution to (0.1). Subsection (1.2) addresses the case when SDEs do not have explicit solutions. We define some schemes which approximate the SDE such as Euler-Maruyama and Milistein ones and we interpret these schemes numerically, graphically and mathematically when the SDE has an explicit solution (e.g Geometric Brownian Motion). Finally, in Subsection 1.3, we introduce the meaning of spread density and scale function to characterize each diffusion process.

In section 2, we proceed graphically to show the behavior of a diffusion process near the boundaries of the state space. Moreover, we know that the drift and volatility functions of the diffusion process are defined on the state space $\mathcal{I}$, but numerically the diffusion can get outside from the state space, which leads to a jump at the boundary explained by the asymptotic behavior of speed and measure function at the endpoints of the state space. But since the diffusion process is a Markov one with continuous paths, the only possible further behavior after reaching the boundary, is absorption. The question which arises now is: when are the boundaries attainable or attracting? Subsection 2.1 is an attempt to answer the question. This is followed by subsection 2.2 in which we recall all types of modern boundary behavior (e.g Wright-Fisher boundary behavior) using and shedding light on some probabilistic significance of the objects S, Σ, M and N.

In section 3, we give all possible initializing state space, where the random variable X_0 lives in the state $\mathbb{I} \subseteq \mathcal{I}$ with left boundary x_l and right one x_r using some measure which characterize the speed of the diffusion process outside and inside the state space. Thereafter, we study the boundary behavior of the diffusion process, starting in one of the following intervals $[x_l, x_r]$, $[x_l, x_r[$, $]x_l, x_r]$ and $]x_l, x_r[$, which are presented on different tables with appropriate conditions.

1 Generalities on diffusion processes

In this section, we present the model, the basic assumptions, and we introduce notation that will be used throughout the paper. Several graphical illustrations of stochastic diffusion processes are included.

1.1 Model, assumptions and notation

In the sequel, we suppose that the process X is a time-homogeneous linear diffusion with dynamics (0.1). If killing is allowed at some time ζ, then the dynamics in (0.1) are valid for $0 \le t < \zeta$. The process X can only be killed at the endpoint of $\mathbb{I}$ which belong to $\mathcal{I}$.

Introduction, existence and uniqueness result The diffusion processes are intensively used nowadays for modeling many phenomena. From the mathematical point of view there exist continuous Markov processes satisfying the SDE (0.1), where the drift coefficient $\mu(x)$ will have the interpretation:

$$\mu(x) := \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}_x \left[X_h - x\right], \tag{1.1}$$

and the diffusion coefficient $\sigma(x)$ satisfies :

$$\sigma^2(x) := \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}_x \left[(X_h - x)^2\right], \tag{1.2}$$

where $\mathbb{E}_x$ denote the conditional expectation such that $X_0 = x \in \mathbb{I} \subseteq \mathcal{I}$. There are several approaches in seeking the solution of a stochastic differential equation.

Uniqueness of Itô solution for the SDE (0.1). We elaborate a number of conditions guaranteeing existence and uniqueness for solutions of the stochastic differential equation. We shall work under the following condition :

Growth Condition: There exists a positive constant K independent of $x \in (-\infty, \infty)$ such that

$$\mu^2(x) + \sigma^2(x) \le K^2(1 + x^2), \quad x \in (-\infty, \infty). \tag{1.3}$$

Global Lipschitz Conditions: There exists a constant $K > 0$ independent of $x \in (-\infty, \infty)$ and $y \in (-\infty, \infty)$ such that

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \le K |x - y|. \tag{1.4}$$

Definition 1. A strong solution X of the stochastic differential equation (0.1) is a stochastic process $(X_t)_{t \ge 0}$ satisfying:

(A) X is adapted to the natural filtration of Brownian motion.

(B) X is a continuous process.

(C) $\mathbb{P}(X_0 = x) = 1$; for all $x \in \mathbb{I}$.

(D) $\mathbb{P}\left(\int_0^t \mu(X_s) + \sigma^2(X_s) ds < \infty\right) = 1$ holds for all $t \ge 0$.

(E) With probability one we have

$$X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s, \tag{1.5}$$

We then have the following:

Theorem 1.1. ***(Theorem 2.5., [12])***
Suppose that the coefficients $\mu(.)$ and $\sigma(.)$ are locally Lipschitz continuous in the space variable; i.e., for every integer $n \ge 1$ there exist a constant $K_n > 0$ such that for every $|x| < n$ and $|y| < n$

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \le K_n |x - y|.$$

Then strong uniqueness hold for equation (1.5).

Remark 1.2. It is worth noting that even for ordinary differential equations, a local Lipschitz condition is not sufficient to guarantee global existence of a solution. For example, the unique (due to Theorem 2.5) solution to the equation :

$$X_t = 1 + \int_0^t X_s^2 ds$$

is $X_t = \frac{1}{1-t}$ which "exploses" as $t \uparrow 1$.

We thus impose stronger conditions in order to obtain an existence result.

Theorem 1.3. ***(Theorem 2.9., [12])*** *Suppose that the coefficients $\mu(.)$ and $\sigma(.)$ satisfies the global Lipschitz and linear growth conditions respectively defined by Equation (1.3) and (1.4), let ξ be a random variable, independent of the Brownian motion $(B_t)_{t\geq 0}$ and with finite second moment. Then there exists a continuous, adapted process $\{X_t, \mathcal{F}_t\ 0 \leq t < \infty\}$ which is a strong solution of Equation (1.5) relative to $(B_t)_{t\geq 0}$, with initial value ξ, more precisely this process is square-integrable : for every $T > 0$, there exists a constant $C > 0$ depending only on K and T, such that :*

$$\mathbb{E}\left[|X_t|^2\right] \leq C\left(1 + \mathbb{E}\left[|\xi|^2\right]\right)e^{Ct}; \quad 0 \leq t \leq T$$

1.2 Preliminary comments on approximation

Many systems of SDEs do not have a (known) analytic solution, so it is necessary to solve these systems numerically. To develop an approximate solution on the interval $[0, T]$, let assign a grid of points

$$0 \equiv t_{n_0} < t_{n_1} < \ldots < t_{n-1} < t_n \equiv T$$

with step length $\Delta = t_k - t_{k-1} = \frac{T}{n}$, we compute the approximate solution as follows:

Euler-Maruyama approximation The Euler-Maruyama method (Maruyama, 1955) is the analogue of the Euler method for ordinary differential equations, it is defined by the following recursion:

$$\begin{aligned}
\hat{X}^n_{t_0} &= X_0, \\
\hat{X}^n_{t_k} &= \hat{X}^n_{t_{k-1}} + \mu\left(\hat{X}^n_{t_{k-1}}\right)\frac{T}{n} + \sigma\left(\hat{X}^n_{t_{k-1}}\right)\sqrt{\frac{T}{n}}\xi_k, \\
\hat{X}^n_t &= \hat{X}^n_{t_{k-1}} \quad t \in [t_{k-1}, t_k[, \quad k = 1, \ldots, n,
\end{aligned}$$

where $(\xi_k)_{k\geq 1}$ is i.i.d. sequence of (0,1)-Gaussian random variables. The following algorithm can be used to simulate a trajectory of the process X using the Euler algorithm instead of the simulation of the stochastic process:

input $X_0, T > 0, n$
$\widehat{X}[1] = X_0$
for k from 1 to $n-1$
generate $\xi \sim \mathcal{N}(0, 1)$
set $\widehat{X}[k+1] = \widehat{X}[k] + \mu(\widehat{X}[k])\frac{T}{n} + \sigma(\widehat{X}[k])\sqrt{\frac{T}{n}}\xi$
end for
return: $\widehat{X}[1], \widehat{X}[2], \ldots, \widehat{X}[n]$

The Matlab code of Euler-Maruyama is given as follows for particular choice of the diffusion coefficients functions $\mu(.)$ and $\sigma(.)$ defined on the state space $\mathcal{I}$.

```
clear all;
clc
% dXt = mu(X_t)dt + sigma(X_t)dBt for t in [0,T]$
T=200;
m = 1; %number of path to simulate
N =1000; % number of sub-intervals of [0,T]
Delta_t = T/N; % the mesh
Z = normrnd(0,1,N,m);
x0=2; %The initial value
B = zeros(N+1,m); %initialization of brownian motion
times = zeros(N,1); %initialization of times
wX=x0*ones(N,m); %initial vector
for n = 1 :  N-1
B(n+1,:) = B(n,:) + sqrt(Delta_t)*Z(n,:);
wX(n+1,:)=wX(n,:)+ mu(wX(n,:))*Delta_t ...
+sigma(wX(n,:)).*(B(n+1,:) - B(n,:));
times(n+1,1) = times(n,1) + Delta_t;
end
```

As example, we take the case of Geometric Brownian Motion (GBM) more over $(X_t)_{t\geq 0}$ satisfy the following SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{1.6}$$

which admitting an explicit solution

$$X_t = X_s \exp\left\{(\mu - \frac{1}{2}\sigma^2)(t-s) + (B_t - B_s)\right\}$$

for all $t > s$ and for all given constants $\mu, \sigma > 0$.

Graphically Figure (1) compares the approximation path of Geometric motion using the Euler Maruyama scheme, with its explicit solution.

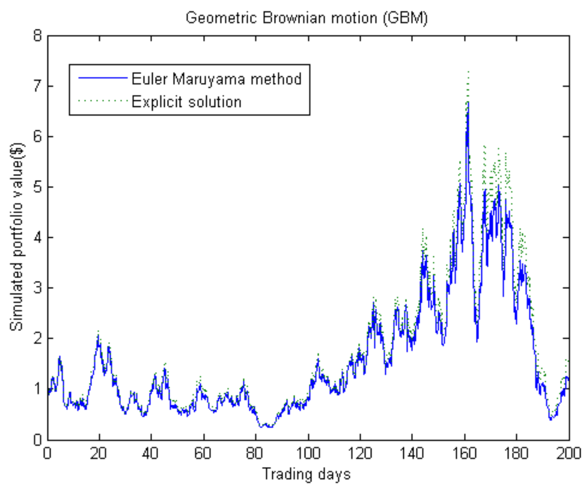

Fig. 1: Euler Maruyama Scheme for numerical approximation of Geometric Brownian Motion, with it's explicit solution.

To study the convergence of numerical schemes, Let $X^{(\Delta)}_{t_n}$ be an approximation of X_{t_n} for a solution X_t of SDE (0.1), and let $\Delta = \frac{T}{n}$. In the literature one mainly considers average error criteria

Strong approximation of order γ

$$\sup_{k=1\ldots n} \mathbb{E}(|\ X_{t_k} - X^{(\Delta)}_{t_k}\ |) \leq K_T \Delta^\gamma, \tag{1.7}$$

Weak approximation of order β

$$|\ \mathbb{E}(g(X_T)) - \mathbb{E}(g(X^{(\Delta)}_{t_n}))\ | \leq K_{g,T}\Delta^\beta, \tag{1.8}$$

for smooth test functions $g(.) : \mathbb{R} \to \mathbb{R}$

Theorem 1.4. *(see [14]) Let assume that the drift and diffusion coeffition satisfy* (1.3) *and* (1.4)*, and let us assume also* $\mathbb{E}[X_0^2] < \infty$*. Then the Euler Maruyama Scheme with apprixation* $\hat{X}^n_{t_n}$ *of* X_{t_n} *for a solution* X_t *of SDE* (0.1) *converges strongly with order* $\gamma = \frac{1}{2}$ *and weakly with order* $\beta = 1$*.*

Milstein approximation The Milstein method is a generalization of the Euler-Maruyama one. The Milstein method approximates the derivative of the function $\sigma(X_t)$ in order to reduce the error to just Δ. It is similar to the Euler-Maruyama except in the computation of $\hat{X}^n_{t_n}$ above. Instead we compute $\bar{X}^n_{t_n}$ as:

$$\begin{aligned} \bar{X}^n_{t_0} &= X_0, \\ \bar{X}^n_{t_k} &= \bar{X}^n_{t_{k-1}} + \mu\left(\bar{X}^n_{t_{k-1}}\right)\frac{T}{n} + \sigma\left(\bar{X}^n_{t_{k-1}}\right)\sqrt{\frac{T}{n}}\xi_k \\ &\quad + \frac{1}{2}\sigma(\bar{X}^n_{t_{k-1}})\sigma'(\bar{X}^n_{t_{k-1}})((\xi_k)^2 - t_{k-1}), \\ \bar{X}^n_t &= \bar{X}^n_{t_{k-1}} \quad t \in [t_{k-1}, t_k[, \quad k = 1, \ldots, n. \end{aligned}$$

To simulate a trajectory of the process X solution of (0.1) using the Milistein algorithm instead of the simulation of the stochastic process, we use the following algorithm:

input X_0, $T > 0$, $n > 0$
$\bar{X}[1] = X_0$
for $k = 1$ to $n - 1$
generate $\xi \sim N(0, 1)$
set $\bar{X}[k+1] = \bar{X}[k] + \mu(\bar{X}[k])h + \sigma(\bar{X}[k])\sqrt{\frac{T}{n}}\xi + 1/2\sigma(\bar{X}[k])\sigma'(\bar{X}[k])((\sqrt{\frac{T}{n}}\xi)^2 - \frac{T}{n})$
end for
return: $\bar{X}[1], \bar{X}[2], \ldots, \bar{X}[n]$

The Matlab code is given as follows, for a particular choice of the diffusion coefficients $\mu(.)$ and $\sigma(.)$:

```
clear all;
clc
% dXt = mu(X_t)dt + sigma(X_t)dBt for t in [0,T]$
T=1;
m = 1; %number of path to simulate
N =1000; % number of sub-intervals of [0,T]
Delta_t = T/N; % the mesh
Z = normrnd(0,1,N,m);
x0=2; %The initial value
B = zeros(N+1,m); %initialization of brownian motion
times = zeros(m+1,1); %initialization of times
mX=x0*ones(N+1,m);
for n = 1 : N
B(n+1,:) = B(n,:) + sqrt(Delta_t)*Z(n,:);
DBn=B(n+1,:)-B(n,:);
mX(n+1,:)=mX(n,:)+ mu(mX(n,:))*Delta_t ...
+sigma(mX(n,:)).*DBn ...
+0.5 sigma(mX(n,:))*sigmaâĂŽ(mX(n,:))(DBn^2-Delta_t);
times(n+1,1) = times(n,1) + Delta_t;
end
```

We compare graphically the approximation path of Geometric motion using the Milstein scheme, with its explicit solution (see Figure (1)).

Theorem 1.5. *Assume that the drift and diffusion coeffition satisfy* (1.3) *and* (1.4)*. Assume also that* $\mathbb{E}[X_0^2] < \infty$*. Then the Milstein scheme with apprixmation* $\bar{X}^n_{t_n}$ *of* X_{t_n} *for a solution* X_t *of SDE* (0.1) *converges strongly with order* $\gamma = 1$ *and weakly with order* $\beta = 1$*.*

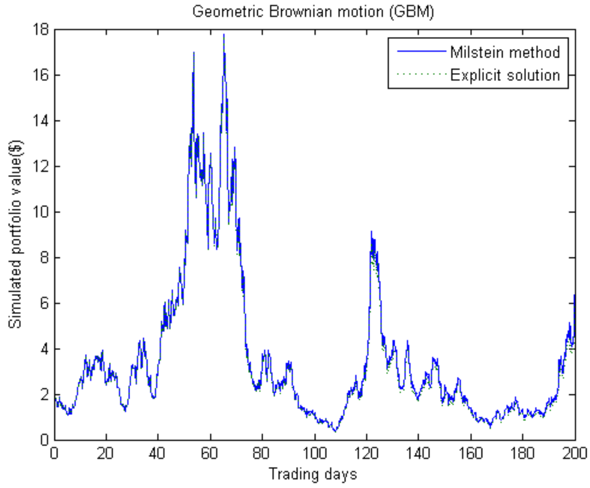

Fig. 2: Milstein Scheme for numerical approximation of Geometric Brownian Motion (GBM), with it's explicit solution.

Proof. see [14] □

1.3 Meaning of speed density and scale function

Our aims in this section is to introduce and explain the meaning of speed density and scale function, which requires assumptions of nondegeneracy and local integrability:

(ND) $$\sigma^2(x) > 0 \quad \forall x \in \mathbb{R},$$

(IC) $$\int_{x-\epsilon}^{x+\epsilon} \frac{1 + |\mu(y)|}{\sigma^2(y)} dy < \infty.$$

We begin with stopping theory which is frequently used to generalize certain properties of stochastic processes.

Hitting times of points and sets play a fundamental role in the study of one-dimensional diffusion processes. Formally we define the hitting time of the process $\{X_t; 0 < t < \zeta\}$ to the level z by

$$\tau_z = \begin{cases} \infty & \text{if } X_t \neq z, \forall\, 0 < t < \zeta \\ \inf\{t > 0; X_t = z\} & \text{otherwise.} \end{cases} \tag{1.9}$$

(See Figure (3))
Let $a < l < r < b$, so let us use the notation

$$\tau^* = \tau_{l,r} = \min\{\tau_l, \tau_r\} = \tau_l \wedge \tau_r$$

for the hitting time to l or r, the first time $X_t = l$ or $X_t = r$. For processes starting at $X_0 = x$ in (a, b), this is the same as the exit time of the interval (a, b):

$$\tau_{a,b} = \inf\{t > 0; X_t \notin (a, b)\}, \; X_0 = x \text{ in } (a, b).$$

A one-dimensional diffusion process X with dynamics (0.1) is called regular, if for any $x \in int(\mathcal{I})$ and $y \in \mathcal{I}$ we have $\mathbb{P}_x(\tau_y < +\infty) > 0$. We concentrate ourselves

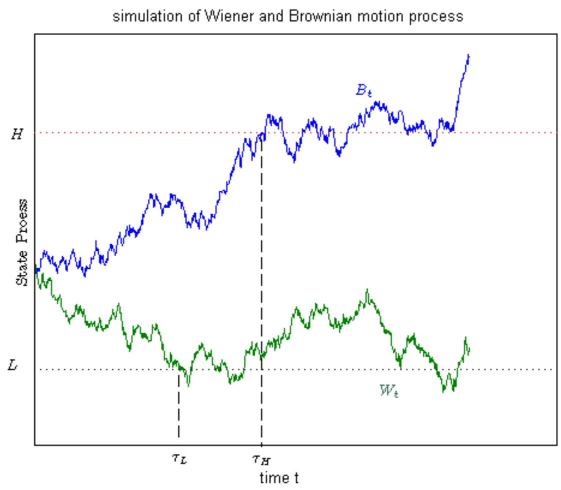

Fig. 3: Typical paths depicting first hitting times.

on the resolution of the following two problems for all $l < x < r$.

Problem 1

$$u(x) = \mathbb{P}_x\left[\tau_l < \tau_r\right]. \quad (1.10)$$

Find the probability that the process reaches l before r given that $X_0 = x$.

Problem 2

$$v(x) = \mathbb{E}_x\left[\tau^*\right]. \quad (1.11)$$

Find the mean time taken by the process to reach l or r given that $X_0 = x$
Using the same meaning as [13], we find that these functions (1.10) and (1.11) satisfies respectively the following ordinary differential Equations:

Equation 1

$$\mu(x)\frac{du}{dx} + \frac{1}{2}\frac{d^2u}{dx^2} = 0 \quad l < x < r, \quad u(l) = 1,\ u(r) = 0.$$

Equation 2

$$\mu(x)\frac{dv}{dx} + \frac{1}{2}\frac{d^2v}{dx^2} = -1 \quad l < x < r, \quad v(l) = v(r) = 0.$$

Following [13] the explicit solution to these ordinary differential equations (1.10) and (1.11) are respectively given by:

Solution 1

$$u(x) = \frac{S(x) - S(l)}{S(r) - S(l)} \quad \text{for } l < x < r.$$

Under Assumptions (**ND**) and (**IC**), $S(x)$ is a scale function with explicit expression:

$$S(x) = \int_c^x s(y)dy,\ s(y) = \exp\left\{-\int_c^y \frac{2\mu(z)}{\sigma^2(z)}dz\right\}. \quad (1.12)$$

Solution 2

$$v(x) = 2\left\{u(x)\int_x^r [S(r) - S(y)]m(y)dy + (1 - u(x))\int_l^x [S(x) - S(l)]m(y)dy\right\}$$

for $l < x < r$, where $m(x) = \frac{1}{\sigma^2(x)s(x)}$ is the speed density.

2 Boundary classification of diffusion process

In this section, we are interested in studying the boundary classification of the regular linear diffusion process $(X_t)_{t\geq 0}$ defined by dynamics (0.1) living in state space $\mathcal{I}$ with right boundary "b" and the left "a" such that $a < b$.

To study the boundary classification of a given one-dimensional diffusion process, we define the following measures:

- The scale measure is the function $\mathsf{S}[J]$ of closed intervals $J = [c, d] \subset (a, b)$ defined by:

  $$\mathsf{S}[J] = \mathsf{S}[c, d] = S(d) - S(c).$$

  For $a < c < d < b$, we have $0 < \mathsf{S}[c, d] < \infty$, and

  $$\mathsf{S}[c, d] = \mathsf{S}[c, x] + \mathsf{S}[x, d], \quad \text{for } a < c < x < d < b, \quad (2.1)$$

  which measures the time the diffusion process takes to reach d before c

- The speed measure M is induced by the speed density $m(x) = \dfrac{1}{\sigma^2(x)s(x)}$, where

  $$\mathsf{M}[J] = \mathsf{M}[c, d] = \int_c^d m(x)dx \quad J = [c, d] \subset (a, b).$$

  $\mathsf{M}[J]$ is positive and finite for $J = [c, d] \subset (a, b)$. Then we can define

  $$\mathsf{M}(a, r] = \lim_{l \downarrow a} \mathsf{M}[l, r] \leq \infty, \quad a < r < b, \quad (2.2)$$

  which measures the speed of the diffusion process near to a.

- Now we assume that $\mathsf{S}(a, x] < \infty$ for all x in (a, b). Then we can define

  $$\begin{aligned}\Sigma(a, x] &= \lim_{l \downarrow a}\int_l^x \mathsf{S}[l, y]d\mathsf{M}(y) \\ &= \int_a^x \mathsf{S}[a, y]d\mathsf{M}(y) \\ &= \int_a^x\left(\int_a^y s(z)dz\right)m(y)dy \\ &= \int_a^x\left(\int_z^x m(y)dy\right)s(z)dz \\ &= \int_a^x \mathsf{M}[z, x]d\mathsf{S}(z), \quad (2.3)\end{aligned}$$

which measures the time the diffusion process takes to reach the boundary "a" from an interior point x of the state space.

- Now we define the following measure of time:

$$\mathsf{N}(a,x] = \int_a^x \mathsf{S}[y,x]d\mathsf{M}(y) = \int_a^x \mathsf{M}(a,y]d\mathsf{S}(y), \qquad (2.4)$$

which measures the time the diffusion process takes to reach an interior point x from (a,b) starting at the boundary "a".

2.1 Classical boundary classification

In this paragraph we make a short review on classical classifications of possible behavior near to the boundaries "a" and "b". We take care of the left boundary "a", the right being entirely similar.

Attracting Boundary It follows from the non-negativity of the measure S and (1.12) that $\mathsf{S}[l,r]$ is decreasing in l for fixed r and that therefore we may define $\mathsf{S}(a,r] \leq \infty$ by:

$$\mathsf{S}(a,r] = \lim_{l \downarrow a} \mathsf{S}[l,r] \leq \infty, \qquad a < r < b. \qquad (2.5)$$

which characterizes the attractiveness or repulsiveness of the left boundary. If $[l,r] \subset (a,b)$, then we get $0 \leq \mathsf{S}[l,r] < \infty$ and we have

$$\mathsf{S}(a,r] = \infty \text{ for some } r \in (a,b)$$
$$\text{If and only if } \mathsf{S}(a,r] = \infty \text{ for all } r \in (a,b).$$

After defining properties of the scale function, which requires the following lemma.

Lemma 2.1.

- *Suppose* $\mathsf{S}(a,r] < \infty$*, for any* $r \in (a,b)$*. Then*
$$\mathbb{P}_x(\tau_a < \tau_r) > 0 \quad \textit{for all } a < x < r < b.$$
- *Suppose* $\mathsf{S}(a,r] = \infty$*, for any* $r \in (a,b)$*. Then*
$$\mathbb{P}_x(\tau_a < \tau_r) = 0 \quad \textit{for all } a < x < r < b.$$

Proof. We have

$$\mathbb{P}_x(\tau_a < \tau_r) = \lim_{l \downarrow a} \frac{S(r) - S(x)}{S(r) - S(l)}$$

Since the scale function is increasing, it follows that $S(r) - S(x) > 0$ and we obtain the same result for $\lim_{l \downarrow a} \mathsf{S}[l,r] > 0$ which completes the proof of the first point.
Now since $S(r) - S(x) < \infty$ for all $a < x < r < b$ and $\lim_{l \downarrow a} \mathsf{S}[l,r] = \infty$, it follows that

$$\mathbb{P}_x(\tau_a < \tau_r) = \lim_{l \downarrow a} \frac{S(r) - S(x)}{S(r) - S(l)} = 0.$$

□

In view of Lemma (2.1), we have the following definition :

Definition 2. The left boundary "a" is attracting if $\mathsf{S}(a,x] < \infty$ and this criterion applies independently of x in (a,b).

Example 2.2. Let us consider a Brownian motion with negative drift $dX_t = \mu dt + \sigma dB_t$ where $\mu < 0$. Then $\mathsf{S}[a,x] = e^{-2\mu x} - e^{-2\mu a} \to e^{-2\mu x} < \infty$ as $a \to -\infty$, thus the boundary $l = -\infty$ is an attracting boundary. (see Figure(4)).

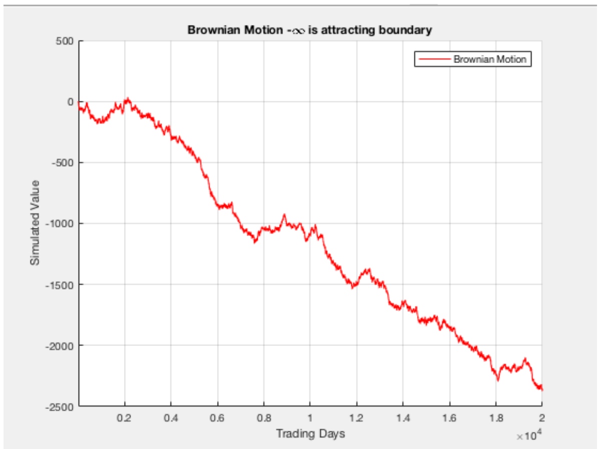

Fig. 4: Brownian motion with negative drift, the boundary $l = -\infty$ is attracting

Atainablte Boundary Note that a boundary "a" is attainable when the process $(X_t)_{t\geq 0}$ may reach "a" almost surely in finite time, if we start the process from interior point x of the state space.

Lemma 2.3. *Let* $a < x < r < b$*. and assume that "a" is an attracting boundary. Then we get the following equivalence :*

(i) $\mathbb{P}_x(\tau_a < \infty) > 0$

(ii) $\mathbb{E}_x[\tau_a \wedge \tau_r] < \infty$

(iii) $\Sigma(a,r] < +\infty$

Proof. Let us prove that $(ii) \Leftrightarrow (iii)$.
We have for $l < x < r$

$$\mathbb{E}_x[\tau_l \wedge \tau_r] = 2\left\{u(x)\int_x^r [S(r) - S(y)]m(y)dy + (1 - u(x))\int_l^x [S(x) - S(l)]m(y)dy\right\} = 2\left\{u(x)\Sigma[x,r) + (1 - u(x))\Sigma(l,x]\right\}.$$

Since "a" is an attracting boundary $\lim_{l \downarrow a} \mathbb{P}_x(\tau_r < \tau_l) < \infty$, we get then the following equivalence :

$$\lim_{l \downarrow a} \mathbb{E}_x[\tau_l \wedge \tau_r] < \infty \quad \Leftrightarrow \quad \lim_{l \downarrow a} \Sigma[l,x].$$

Let us now prove that $(ii) \Leftrightarrow (i)$.
$(ii) \Rightarrow (i)$.

Assume first that $\mathbb{E}_x[\tau_a \wedge \tau_r] < \infty$. Then, $\tau_a \wedge \tau_r < \infty$ a.s. By Lemma (2.1) as "a" is atractive, we have $\mathbb{P}_x(\tau_a < \tau_r) > 0$, therefore

$$\mathbb{P}_x(\tau_a < \infty) \geq \mathbb{P}_x(\tau_a < \tau_r) > 0.$$

$(ii) \Leftarrow (i)$.
Suppose now that $\mathbb{P}_x(\tau_a < \infty) > 0$. Then there exists $t > 0$ for which $\mathbb{P}_x(\tau_a \leq t) = \alpha > 0$. We can use following property for all $y \in (a, x]$

$$\mathbb{P}_y(\tau_a \leq t) \geq \mathbb{P}_x(\tau_a \leq t) = \alpha > 0.$$

Moreover, we get

$$\alpha \leq \mathbb{P}_y(\tau_a \leq t) \leq \mathbb{P}_y(\tau_a \wedge \tau_x \leq t)$$

It follows that

$$\sup_{y\in(a,x]} \mathbb{P}_y(\tau_a \wedge \tau_x > t) \leq 1 - \alpha < 1$$

Using the strong Markov propriety, we get for all $n \geq 1$

$$\mathbb{P}_x(\tau_a \wedge \tau_x > nt) \leq \sup_{y\in(a,x]} \mathbb{P}_y(\tau_a \wedge \tau_x > nt) \leq (1-\alpha)^n.$$

Then we derive $\mathbb{E}_x[\tau_a \wedge \tau_r] < \frac{1}{\alpha} < \infty$.
But since $(ii) \Leftrightarrow (iii)$ we obtain therefore the equivalance $(i) \Leftrightarrow (iii)$.

For details on computations at this step, the reader can consult the paper [1] □

In view of Lemma (2.3), we have the following definition.

Definition 3.

i) The boundary "a" is **attainable** if $\Sigma(a, x] < \infty$, for all $x \in (a, b)$

ii) The boundary "a" is **unattainable** if $\Sigma(a, x] = \infty$, for all $x \in (a, b)$

We can show that $\mathsf{S}(a, x] < \infty$ whenever $\Sigma(a, x] < \infty$, and hence, if "a" is attainable, then "a" is attracting. An unattainable boundary may or may not be attracting.

Example 2.4. The key example of interest to us in this paper is the CIR model with SDE

$$dr_t = -r_t\,dt + \sqrt{2r_t}\,dB_t.$$

We can prove that $S(x) = e^x - 1$, $m(x) = \frac{1}{2xe^x}$ and $s(x) = e^x$. It follows that $\mathsf{S}(0, x] = 1 - e^x < \infty$, then 0 is attracting boundary. Moreover, $\Sigma(0, x] = \lim_{l\downarrow 0} \dfrac{e^l - 1}{2ye^l} = 1 < \infty$, thus 0 is attainable boundary. (see Figure(5)).

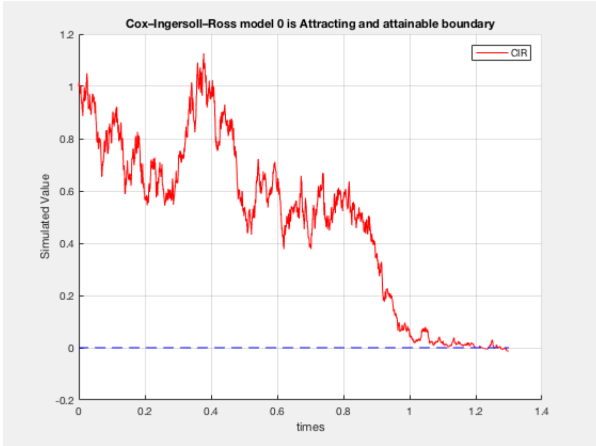

Fig. 5: Cox-Ingersoll-Ross model, the boundary $l = 0$ is attracting and attainable

2.2 Modern boundary classification

Following [13], the boundary classification are described as follow :

1. Regular boundary: the process can either reach the boundary from an interior point, or reach an interior point from the boundary.
   Moreover "a" is an regular boundary if and only if $\mathsf{S}(a, x] < \infty$, $\mathsf{M}(a, x] < \infty$, $\Sigma(a, x] < \infty$, and $\mathsf{N}(a, x] < \infty$.

2. Exit boundary: the process can reach the boundary from an interior point, but cannot reach an interior point from the boundary.
   Moreover "a" is an exit boundary if and only if $\mathsf{S}(a, x] < \infty$, $\mathsf{M}(a, x] = \infty$, $\Sigma(a, x] < \infty$, and $\mathsf{N}(a, x] = \infty$.

3. Entrance boundary: the process can reach an interior point from the boundary, but cannot reach the boundary from an interior point.
   Moreover "a" is an Entrance boundary if and only if $\mathsf{S}(a, x] = \infty$, $\mathsf{M}(a, x] < \infty$, $\Sigma(a, x] = \infty$, and $\mathsf{N}(a, x] < \infty$.

4. Natural boundary: the process cannot reach the boundary from an interior point, nor can it reach an interior point from the boundary.
   Moreover "a" is a natural boundary if and only if $\Sigma(a, x] = \infty$ and $\mathsf{N}(a, x] = \infty$.

For more interpretation, table (1) lists the 6 possible combinations of assignments of finite or infinite values to the four quantities $S(a, x]$, $\Sigma(a, x]$, $N(a, x]$, and $\mathsf{M}(a, x]$, and labeled differently by different authors. William Feller introduced the original classification, adhered to by most American probabilists. The Russian school uses a slightly different formalization. Both groupings have their merits and are now juxtaposed.

Numerical representation of regular, Natural, entrance and exit boundary The Wright-Fisher family

Tab. 1: The terminology of Feller and Russian boundary classification schemes (see [13], Table 6.2)

<table>
<tr><th colspan="4">Criteria</th><th colspan="4">Terminologies</th></tr>
<tr><th>$\mathsf{S}(a,x]$</th><th>$\mathsf{M}(a,x]$</th><th>$\Sigma(a,x]$</th><th>$\mathsf{N}(a,x]$</th><th>Feller</th><th colspan="3">Gikhman and Skorokhod</th></tr>
<tr><td>$< \infty$</td><td>$< \infty$</td><td>$< \infty$</td><td>$< \infty$</td><td colspan="2">regular</td><td rowspan="3">attracting</td><td rowspan="2">attainable</td></tr>
<tr><td>$< \infty$</td><td>$= \infty$</td><td>$< \infty$</td><td>$= \infty$</td><td>exit</td><td>absorbing</td></tr>
<tr><td>$< \infty$</td><td>$= \infty$</td><td>$= \infty$</td><td>$= \infty$</td><td>natural</td><td>attracting-unattainable</td><td rowspan="4">unattainable</td></tr>
<tr><td>$= \infty$</td><td>$< \infty$</td><td>$= \infty$</td><td>$= \infty$</td><td>$\Sigma(a,x] = \infty$</td><td rowspan="3">natural
($\mathsf{S}(a,x] = \infty$)</td><td rowspan="3">no-attracting</td></tr>
<tr><td>$= \infty$</td><td>$= \infty$</td><td>$= \infty$</td><td>$= \infty$</td><td>$\mathsf{N}(a,x] = \infty$</td></tr>
<tr><td>$= \infty$</td><td>$< \infty$</td><td>$= \infty$</td><td>$< \infty$</td><td>Entrance</td></tr>
</table>

of diffusion processes is a class of evolutionary models widely used in population genetics, with applications also in finance and Bayesian statistics. Simulation and inference from these diffusions is therefore of widespread interest. However, simulating a Wright-Fisher diffusion is difficult because the process can comes out from the domain of definition cause to Gaussian Perturbation.
In this part, we develop a new figure where the process jumps at its boundaries or starting from an exterior point of state space (e.g. Exit and Entrance boundaries)
Let a Wright-Fisher diffusion process with dynamic :

$$dX_t = \mu(X_t)dt + \sqrt{X_t(1-X_t)}dB_t, \quad X_0 = x_0,\ t \in [0,T]. \tag{2.6}$$

The drift coefficient, $\mu : [0,1] \to \mathbb{R}$, can encompass a variety of evolutionary forces. For example, $\mu(x) = \frac{1}{2}[\theta_1(1-x) - \theta_2 x]$ where θ_1 and θ_2 are the coefficients mutations between the two alleles. Follwing [13] we get

$$0 \begin{cases} \text{is an exit boundary} & for \quad \theta_1 = 0 \\ \text{is a regular boundary} & for \quad 0 < \theta_1 < \frac{1}{2} \\ \text{is an entrance boundary} & for \quad \theta_1 \geq \frac{1}{2} \end{cases}$$

$$1 \begin{cases} \text{is an exit boundary} & for \quad \theta_2 = 0 \\ \text{is a regular boundary} & for \quad 0 < \theta_2 < \frac{1}{2} \\ \text{is an entrance boundary} & for \quad \theta_2 \geq \frac{1}{2} \end{cases}$$

Not that :
The only accessible boundary points are of two different types:

- **The exit boundary:** the process may be absorbed after reaching boundary since it comes out from the state space. But to understand the physical meaning of Exit boundaries, we continue simulation of wright-fisher model after reaching the boundaries.
- **The regular boundary:** in this case the process lives in the state space and can not exit its boundaries.

The only inaccessible boundary points are of two different types:

- **The natural boundary:** the process can not reach the boundaries in finite times.
- **The entrance boundary:** the process cannot reach the entrance boundaries from the interior of state space but the only possible case is to start the Wright-Fisher model from out side of state space to better understand the physical meaning of Entrance boundaries.

To better understand the physical meaning of regular, entrance, exit and natural boundary, we clarify how we can continue the simulation of Wright-Fisher diffusion process. In spite of the inequality $X_t(1-X_t) < 0$ when X reaches 0 and becomes strictly negative and so the square root is no longer defined. Then, how we can continue the simulation of the Wright-Fisher process? The simple idea is to find the primitive of

$$\frac{1}{\sqrt{x(1-x)}}$$

We can prove easily the fact that

$$\int \frac{1}{\sqrt{x(1-x)}}dx = 2\arcsin\left(\sqrt{x}\right) + constant,$$

Using the change of variable $y = \sqrt{x}$. By applying Ito's Lemma for $f(x) = 2\arcsin\left(\sqrt{x}\right)$ we get

$$\frac{d\Big(f(x)\Big)}{dx} = \frac{1}{\sqrt{x(1-x)}},$$

$$\frac{d^2\Big(f(x)\Big)}{dx^2} = \frac{2x-1}{2(x(1-x))^{\frac{3}{2}}}.$$

Substituting the expression for $Y_t = 2\arcsin\left(\sqrt{X_t}\right)$, we get

$$\begin{aligned} dY_t &= f'(X_t)dX_t + \frac{1}{2}X_t(1-X_t)f''(X_t)dt \\ &= dB_t + \frac{2X_t - 1}{4\sqrt{X_t(1-X_t)}}dt. \end{aligned}$$

We have $Y_t = 2\arcsin\left(\sqrt{X_t}\right)$ so $X_t = \sin^2\left(\frac{Y_t}{2}\right)$. Therefore

$$\begin{aligned} dY_t &= dB_t + \frac{2\sin^2\left(\frac{Y_t}{2}\right) - 1}{4\sqrt{\sin^2\left(\frac{Y_t}{2}\right)\cos^2\left(\frac{Y_t}{2}\right)}}dt \\ &= -\frac{\cos(Y_t)}{2\left|\sin(Y_t)\right|}dt + dB_t \\ &= -\frac{1}{2}\text{signe}(\sin(Y_t))\cot(Y_t)dt + dB_t \end{aligned}$$

Hence, we can first simulate the Wright-Fisher diffusion process. Next we simulate the new process Y_t using

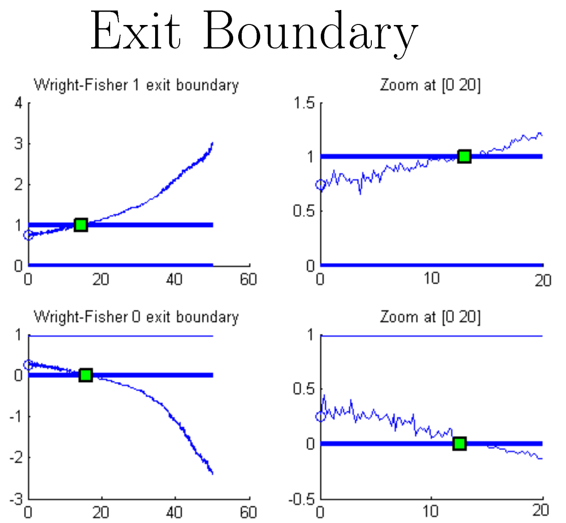

Fig. 6: Wright-Fisher model, $\theta_{=}0, \theta_2 = 0$ the boundary $l = 0$ and $r = 1$ are Exit

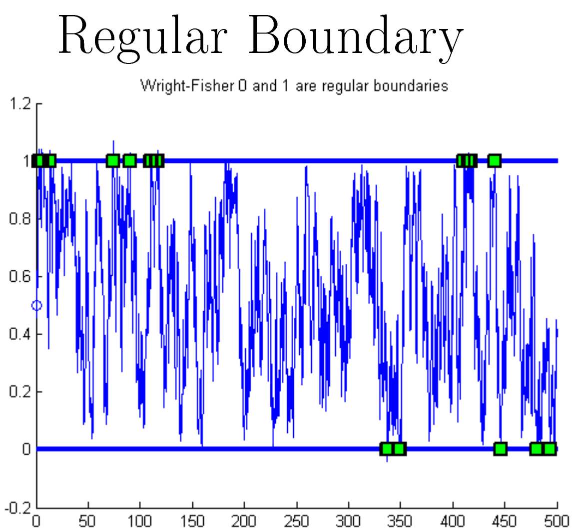

Fig. 7: Wright-Fisher model, $\theta_{=}0.25, \theta_2 = 0.25$ the boundary $l = 0$ and $r = 1$ are regular

3 All possible types of initializing state

We are now interested in the behavior of diffusion process X of Equation (0.1) at the endpoint x_l and x_r of $\mathbb{I}$. By using the previous results, we can see that if an endpoint is included in the state-space $\mathcal{I}$, then the boundary is either "regular or exit". Else if it is not contained in $\mathcal{I}$, then it is either "natural or entrance". Moreover we assume that we will stop the process as soon as it reaches one of the boundary. To cover all possibilities of starting process interval $\mathbb{I}$, moreover to get results in full generality taken account of **Feller**

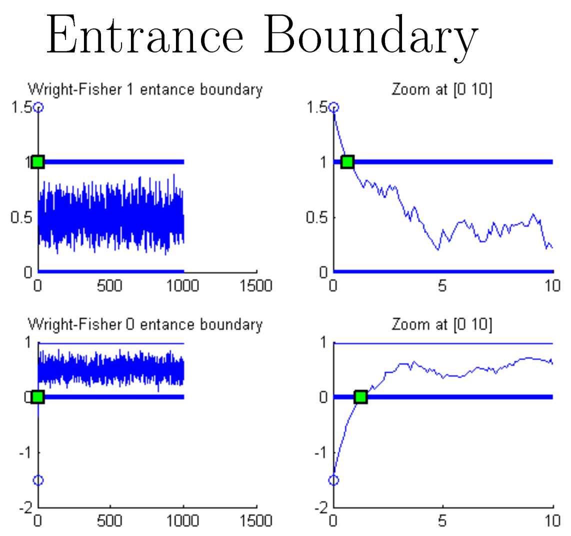

Fig. 8: Wright-Fisher model, $\theta_{=}0.5, \theta_2 = 0.5$ the boundary $l = 0$ and $r = 1$ are Entrance

boundary classification, we will consider the following four cases for the sub-interval $\mathbb{I}$ of the state space $\mathcal{I}$:

Case1

$$\left\{ \begin{array}{c} \text{The boundaries } x_l = c \text{ and } x_r = d \text{ are} \\ \text{"regular" or "exit" :} \\ \mathbb{I} = [c, d], c < d \text{ with } c, d \in \mathcal{I}. \end{array} \right\} \quad (3.1)$$

Case2

$$\left\{ \begin{array}{c} \text{The boundary } x_r = b \text{ is} \\ \text{"regular" or "exit"} \\ \text{and the boundary } x_l = c \text{ is} \\ \text{"natural" or "entrance"} \\ \mathbb{I} = [c, b) \text{ where } \in \mathcal{I}. \end{array} \right\} \quad (3.2)$$

Case3

$$\left\{\begin{array}{c}\text{The boundary } x_l = a \text{ is} \\ \text{"regular" or "exit"} \\ \text{and the boundary } x_r = d \text{ is} \\ \text{"natural" or "entrance" :} \\ \mathbb{I} = (a, d] \text{where } d \in \mathcal{I}.\end{array}\right\} \quad (3.3)$$

$$(3.4)$$

Case4

$$\left\{\begin{array}{c}\text{The boundaries } x_r = b \text{ and } x_l = a \text{ are} \\ \text{"natural" or "entrance" :} \\ \mathbb{I} = (a, b) = \mathcal{I}.\end{array}\right\} \quad (3.5)$$

We are interested now in giving a general condition that covers all cases of subinterval $\mathbb{I} \subseteq \mathcal{I}$ given by (3.1), (3.2), (3.3) and (3.5). The key condition is in scale function "$S(.)$", speed measures "M" and time measures "Σ" and "N". Tables (2), (3),(4) and (5) summarize all possible combination of endpoints for the subinterval $\mathbb{I}$ of the state space $\mathcal{I}$.

Concluding remarks

Diffusion processes are very useful in mathematical modeling apart from their intrinsic interest. We can cite for example the famous Black-Scholes model used in finance and the Wright-Fisher diffusion process intensively used in mathematical biology. But it seems important to proceed graphically to study any boundary behavior, since the nature boundaries has an influence on the resolution of several problems such as standard optimal stopping [10] and multiple stopping [11]. In this paper, we have focused on the modern classification of possible behavior near the boundaries of any interval in which a linear diffusion process $(X_t)_{t\geq 0}$ diffuses. We have shed light on simulation of a one-dimensional diffusion process using discretization techniques and we have presented the simulation algorithms of Euler-Maruyama and Milstein together with several examples. When the diffusion coefficients are linear functions, we have proceeded by direct simulation of the explicit expression. In the last case, a comparison between the three approaches is illustrated numerically. For each possible kind of boundary, we have provided an example of diffusion process with graphical simulation of its behavior, especially at the endpoints of the initialization sub-interval of the state space. The contribution of the paper lies in the following points:

(i) We offer well-organized tables on the nature of a any boundary (closed or open) of a given subinterval of the state space, with appropriate conditions. In this regard, we have compiled tables which cover all types of diffusion with suitable conditions (presented before the paragraph "Numerical representation of regular, Natural, entrance and exit boundary" of the Subsection (2.2)). These results are scattered in the literature. Their gathering in tables facilitates the readers understanding and quick access to the concept of boundary classification at the end of the Subsection (2.2).

(ii) We provide for every possible type of boundary classification, several examples of diffusion with computer simulations illustrated graphically. The simulation code being compiled and presented separately in Section (1).

In conclusion, we believe that these contributions can only be useful to better understand the physical and natural meaning of classification boundary.

We have assumed that the diffusion process must be stopped when it reaches the exit boundary to avoid jump on exit boundary. By Feller boundary classification, there is at least one case for which the scale function is bounded when both boundaries are natural, take the example of a geometric Brownian motion with dynamics

$$dX_t = \mu X_t dt + \sigma X_t dB_t,$$

on the state space is $\mathcal{I} = (0, +\infty)$ and where both lower and upper of boundaries are natural with $2\mu > \sigma^2$ (see Table (4)).

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Tab. 2: All possible combination of endpoint for the interval $[c, b)$ with appropriate conditions

<table>
<tr><th>[c,b)</th><th colspan="4">The boundary "b" is natural</th><th colspan="4">The boundary "b" is entrance</th></tr>
<tr><td rowspan="3">The boundary c is exit</td><td rowspan="3">$\Sigma(c, x] < \infty$
$\tau_c < \infty$</td><td>$\mathsf{S}[x, b) < \infty$
S bounded</td><td>$\mathsf{M}(c, x] = \infty$
$\mathsf{M}[x, b) = \infty$</td><td rowspan="3">$\mathsf{N}(c, x] = \infty$
$\mathsf{N}[x, b) = \infty$</td><td rowspan="3">$\Sigma(c, x] < \infty$
$\tau_c < \infty$</td><td rowspan="6">$\mathsf{S}[x, b) = \infty$
S unbounded</td><td rowspan="3">$\mathsf{M}(c, x] = \infty$
$\mathsf{M}[x, b) < \infty$</td><td rowspan="3">$\mathsf{N}(c, x] = \infty$
$\mathsf{N}[x, b) < \infty$</td></tr>
<tr><td rowspan="2">$\mathsf{S}[x, b) = \infty$
S unbounded</td><td>$\mathsf{M}(c, x] = \infty$
$\mathsf{M}[x, b) < \infty$</td></tr>
<tr><td>$\mathsf{M}(c, x] = \infty$
$\mathsf{M}[x, b) = \infty$</td></tr>
<tr><td rowspan="3">The boundary c is regular</td><td rowspan="3">$\Sigma[x, b) = \infty$
$\tau_b = \infty$</td><td>$\mathsf{S}[x, b) < \infty$
S bounded</td><td>$\mathsf{M}(c, x] < \infty$
$\mathsf{M}[x, b) = \infty$</td><td rowspan="3">$\mathsf{N}(c, x] < \infty$
$\mathsf{N}[x, b) = \infty$</td><td rowspan="3">$\Sigma[x, b) = \infty$
$\tau_b = \infty$</td><td rowspan="3">$\mathsf{M}(c, x] < \infty$
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$\mathsf{N}[x, b) < \infty$</td></tr>
<tr><td rowspan="2">$\mathsf{S}[x, b) = \infty$
S unbounded</td><td>$\mathsf{M}(c, x] < \infty$
$\mathsf{M}[x, b) < \infty$</td></tr>
<tr><td>$\mathsf{M}(c, x] < \infty$
$\mathsf{M}[x, b) = \infty$</td></tr>
</table>

Tab. 3: All possible combination of endpoint for the interval $(a, d]$ with appropriate conditions

<table>
<tr><th>(a,d]</th><th colspan="4">The boundary "a" is natural</th><th colspan="4">The boundary "a" is entrance</th></tr>
<tr><td rowspan="3">The boundary d is exit</td><td rowspan="3">$\Sigma[x, d) < \infty$
$\tau_d < \infty$</td><td>$\mathsf{S}(a, x] < \infty$
S bounded</td><td>$\mathsf{M}[x, d) = \infty$
$\mathsf{M}(a, x] = \infty$</td><td rowspan="3">$\mathsf{N}[x, d] = \infty$
$\mathsf{N}(a, x] = \infty$</td><td rowspan="3">$\Sigma[x, d) < \infty$
$\tau_d < \infty$</td><td rowspan="6">$\mathsf{S}(a, x] = \infty$
S unbounded</td><td rowspan="3">$\mathsf{M}[x, d) = \infty$
$\mathsf{M}(a, x] < \infty$</td><td rowspan="3">$\mathsf{N}[x, d) = \infty$
$\mathsf{N}(a, x] < \infty$</td></tr>
<tr><td rowspan="2">$\mathsf{S}(a, x] = \infty$
S unbounded</td><td>$\mathsf{M}[x, d) = \infty$
$\mathsf{M}(a, x] < \infty$</td></tr>
<tr><td>$\mathsf{M}[x, d) = \infty$
$\mathsf{M}(a, x] = \infty$</td></tr>
<tr><td rowspan="3">The boundary d is regular</td><td rowspan="3">$\Sigma(a, x] = \infty$
$\tau_a = \infty$</td><td>$\mathsf{S}(a, x] < \infty$
S bounded</td><td>$\mathsf{M}[x, d) < \infty$
$\mathsf{M}(a, x] = \infty$</td><td rowspan="3">$\mathsf{N}(x, d] < \infty$
$\mathsf{N}(a, x] = \infty$</td><td rowspan="3">$\Sigma(a, x] = \infty$
$\tau_a = \infty$</td><td rowspan="3">$\mathsf{M}[x, d) < \infty$
$\mathsf{M}(a, x] < \infty$</td><td rowspan="3">$\mathsf{N}[x, d) < \infty$
$\mathsf{N}(a, x] < \infty$</td></tr>
<tr><td rowspan="2">$\mathsf{S}(a, x] = \infty$
S unbounded</td><td>$\mathsf{M}[x, d) < \infty$
$\mathsf{M}(a, x] < \infty$</td></tr>
<tr><td>$\mathsf{M}[x, d) < \infty$
$\mathsf{M}(a, x] = \infty$</td></tr>
</table>

Tab. 4: All possible combination of endpoint for the interval $[c, d]$ with appropriate conditions

<table>
<tr><th>[c,d]</th><th colspan="4">The boundary c is regular</th><th colspan="4">The boundary d is exit</th></tr>
<tr><td>The boundary c is exit</td><td>$\Sigma[x, d) < \infty$
$\tau_d < \infty$</td><td rowspan="2">$\mathsf{S}(c, x] < \infty$
and
$\mathsf{S}[x, d) < \infty$
S is bounded</td><td>$\mathsf{M}[x, d) < \infty$
$\mathsf{M}(c, x] = \infty$</td><td>$\mathsf{N}[x, d) < \infty$
$\mathsf{N}(c, x] = \infty$</td><td>$\Sigma[x, d) < \infty$
$\tau_d < \infty$</td><td rowspan="2">$\mathsf{S}(c, x] < \infty$
and
$\mathsf{S}[x, d) < \infty$
S is bounded</td><td>$\mathsf{M}[x, d) = \infty$
$\mathsf{M}(c, x] = \infty$</td><td>$\mathsf{N}[x, d) = \infty$
$\mathsf{N}(c, x] = \infty$</td></tr>
<tr><td>The boundary c is regular</td><td>$\Sigma(c, x] < \infty$
$\tau_c < \infty$</td><td>$\mathsf{M}(c, x] < \infty$
$\mathsf{M}[x, d) < \infty$</td><td>$\mathsf{N}(c, x] < \infty$
$\mathsf{N}[x, d) < \infty$</td><td>$\Sigma(c, x] < \infty$
$\tau_c < \infty$</td><td>$\mathsf{M}[x, d) = \infty$
$\mathsf{M}(c, x] < \infty$</td><td>$\mathsf{N}[x, d) = \infty$
$\mathsf{N}(c, x] < \infty$</td></tr>
</table>

Tab. 5: All possible combination of terminals of an interval of the form (a, b) with appropriate conditions

(a,b)	the boundary "b" is natural											
"a" natural	$\mathsf{S}(a,x] < \infty$ $\mathsf{S}[x,b) < \infty$ S is bounded	$\mathsf{M}(a,x] = \infty$ $\mathsf{M}[x,b) = \infty$	$\tau_a = \infty$ $\Sigma(a,x] = \infty$ $\tau_b = \infty$ $\Sigma[x,b) = \infty$	$\mathsf{N}(a,x] = \infty$ $\mathsf{N}[x,b) = \infty$	$\mathsf{S}(a,x] < \infty$ $\mathsf{S}[x,b) = \infty$ S is unbounded	$\mathsf{M}(a,x] = \infty$ $\mathsf{M}[x,b) < \infty$	$\tau_a = \infty$ $\Sigma(a,x] = \infty$ $\tau_b = \infty$ $\Sigma[x,b) = \infty$	$\mathsf{N}(a,x] = \infty$ $\mathsf{N}[x,b) = \infty$	$\mathsf{S}(a,x] < \infty$ $\mathsf{S}[x,b) = \infty$ S is unbounded	$\mathsf{M}(a,x] = \infty$ $\mathsf{M}[x,b) = \infty$	$\tau_a = \infty$ $\Sigma(a,x] = \infty$ $\tau_b = \infty$ $\Sigma[x,b) = \infty$	$\mathsf{N}(a,x] = \infty$ $\mathsf{N}[x,b) = \infty$
	$\mathsf{S}(a,x] = \infty$ $\mathsf{S}[x,b) < \infty$ S is unbounded	$\mathsf{M}(a,x] < \infty$ $\mathsf{M}[x,b) = \infty$			$\mathsf{S}(a,x] = \infty$ $\mathsf{S}[x,b) = \infty$ S is unbounded	$\mathsf{M}(a,x] < \infty$ $\mathsf{M}[x,b) < \infty$			$\mathsf{S}(a,x] = \infty$ $\mathsf{S}[x,b) = \infty$ S is unbounded	$\mathsf{M}(a,x] < \infty$ $\mathsf{M}[x,b) = \infty$		
	$\mathsf{S}(a,x] = \infty$ $\mathsf{S}[x,b) < \infty$ S is unbounded	$\mathsf{M}(a,x] = \infty$ $\mathsf{M}[x,b) = \infty$			$\mathsf{S}(a,x] = \infty$ $\mathsf{S}[x,b) = \infty$ S is unbounded	$\mathsf{M}(a,x] = \infty$ $\mathsf{M}[x,b) < \infty$			$\mathsf{S}(a,x] = \infty$ $\mathsf{S}[x,b) = \infty$ S is unbounded	$\mathsf{M}(a,x] = \infty$ $\mathsf{M}[x,b) = \infty$		

(a,b)	The boundary "b" is entrance			
"a" natural	$\mathsf{S}(a,x] < \infty$ and $\mathsf{S}[x,b) = \infty$ S is unbounded	$\mathsf{M}(a,x] = \infty$ $\mathsf{M}[x,b) < \infty$	$\Sigma(a,x] = \infty$ $\tau_a = \infty$ $\Sigma[x,b) = \infty$ $\tau_b = \infty$	$\mathsf{N}(a,x] = \infty$
	$\mathsf{S}(a,x] = \infty$ and $\mathsf{S}[x,b) = \infty$ S is unbounded	$\mathsf{M}(a,x] < \infty$ $\mathsf{M}[x,b) < \infty$		$\mathsf{N}[x,b) < \infty$
		$\mathsf{M}(a,x] = \infty$ $\mathsf{M}[x,b) < \infty$		
"a" entrance		$\mathsf{M}(a,x] < \infty$ $\mathsf{M}[x,b) < \infty$		$\mathsf{N}(a,x] < \infty$ $\mathsf{N}[x,b) < \infty$

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