Graduate J. Math. 2 (2017), 51 – 58

On the commutativity of sums of Toeplitz operators on the Bergman space

Khitam Aqel and Issam Louhichi

Abstract

In this paper, we discuss the commutativity of sums of two quasihomogeneous Toeplitz operators on the Bergman space of the unit disk. Our main result goes in the direction of the conjecture in [9, p. 263].

MSC 2010. Primary: 47B35; Secondary: 47L80.

1 Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$. We denote by L_a^2 the usual unweighted Bergman space, the Hilbert space of analytic functions on $\mathbb{D}$ that are square integrable with respect to the normalized Lebesgue measure $dA(z) = rdr\frac{d\theta}{\pi}$, where (r, θ) are the polar coordinates. L_a^2 is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ and has the set

$$\{\sqrt{n+1}z^n \mid n = 0, 1, 2, \ldots\}$$

as an orthonormal basis. We denote by P the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_a^2, called the Bergman projection. We define on L_a^2 the Toeplitz operator T_h with symbol a bounded function h by $T_h u = P(hu)$ for any $u \in L_a^2$.

One of the most intriguing questions for Toeplitz operators on the Bergman space is: When is the product (in a sense of composition) of two Toeplitz operators commutative? Although a lot of work has been done on this question [2, 3, 5, 6, 7, 9, 11, 13, 15], we still have not reached a complete answer.

The analogous question on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$, which is the subspace of $L^2(\mathbb{T})$ consisting of functions whose negative Fourier coefficients are equal to zero, was elegantly solved by Brown and Halmos in their seminal paper [4]. They proved that for two symbols ϕ and ψ in $L^\infty(\mathbb{T})$, the Toeplitz operators T_ϕ and T_ψ commute if and only if either both symbols are analytic, or both symbols are conjugate analytic, or there are constants α and β such that $\phi = \alpha\psi + \beta$. Unfortunately, and as usual, the situation in the Bergman space is much more complicated, and a similar result to Brown and Halmos theorem could not have been obtained in L_a^2 without additional hypothesis on the symbols [2, Theorem 1]. For several reasons, the results of [4] cannot be simply mimicked in L_a^2. To cite few of them, one can argue that: The orthogonal complement of L_a^2 is much more than $\overline{L_a^2}$; The matrix of a Toeplitz operators in L_a^2 space is not a Toeplitz matrix; In $H^2(\mathbb{T})$, a bounded operator is a Toeplitz if and only if $S^*TS = T$ (where S is the shift operator in $H^2(\mathbb{T})$). This characterization is not anymore true in L_a^2. More precisely, we only have "kind of similar" characterization of Toeplitz operators with bounded harmonic symbols [8]. For more thorough treatments of this subject, the reader might refer to the survey papers [1] and [14].

2 Preliminaries

The main motivation of this paper is the results of [9]. We shall show that under some assumptions if sums of two quasihomogeneous Toeplitz operators commute, then they are equal up to a multiplicative constant. A symbol h is said to be quasihomogeneous of an integer degree p if h can be written as $h(re^{i\theta}) = e^{ip\theta}\omega(r)$, where ω is a radial function in $\mathbb{D}$ i.e., $\omega(z) = \omega(|z|)$. In this case, the associated Toeplitz operator T_h is also called quasihomogeneous Toeplitz operator of degree p. This class of operators got the interest of many people [6, 7, 9, 10, 11, 12, 13] and have been widely studied since.

Consider two bounded quasihomogeneous Toeplitz operators $T_{e^{ip\theta}\phi}$ and $T_{e^{is\theta}\psi}$ with $p, s \in \mathbb{N}$. Assume that these two operators have roots $T_{e^{i\theta}\widetilde{\phi}}$ and $T_{e^{i\theta}\widetilde{\psi}}$ respectively (see [7, 10]) i.e.,

$$T_{e^{ip\theta}\phi} = \left(T_{e^{i\theta}\widetilde{\phi}}\right)^p \text{ and } T_{e^{is\theta}\psi} = \left(T_{e^{i\theta}\widetilde{\psi}}\right)^s.$$

The purpose of this work is to characterize quasihomogeneous Toeplitz operators $T_{e^{im\theta}f}$ and $T_{e^{il\theta}g}$ where $m, l \in \mathbb{N}$, such that

$(H1)$ $T_{e^{im\theta}f} + T_{e^{il\theta}g}$ commutes with $T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi}$,
$(H2)$ $1 \leq p < s,\ 1 \leq m < l,$ and $l + p = m + s$.

Hypothesis $(H1)$ implies that for all $k \geq 0$, we have

$$\left(T_{e^{im\theta}f} + T_{e^{il\theta}g}\right)\left(T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi}\right)(z^k) = \left(T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi}\right)\left(T_{e^{im\theta}f} + T_{e^{il\theta}g}\right)(z^k).$$

Now hypothesis $(H2)$, combined with [9, Remark 2], implies that for all $k \geq 0$ we must have

$$T_{e^{im\theta}f}T_{e^{ip\theta}\phi}(z^k) = T_{e^{ip\theta}\phi}T_{e^{im\theta}f}(z^k), \qquad (2.1)$$

$$T_{e^{il\theta}g}T_{e^{is\theta}\psi}(z^k) = T_{e^{is\theta}\psi}T_{e^{il\theta}g}(z^k), \qquad (2.2)$$

and

$$\left(T_{e^{im\theta}f}T_{e^{is\theta}\psi} + T_{e^{il\theta}g}T_{e^{ip\theta}\phi}\right)(z^k) = \left(T_{e^{is\theta}\psi}T_{e^{im\theta}f} + T_{e^{ip\theta}\phi}T_{e^{il\theta}g}\right)(z^k). \qquad (2.3)$$

Equations (2.1) and (2.2) imply that $T_{e^{im\theta}f}$ (resp. $T_{e^{il\theta}g}$) commutes with $T_{e^{ip\theta}\phi}$ (resp. $T_{e^{is\theta}\psi}$). Therefore, using [9, Proposition 2 and Lemma 2], we obtain that

$$T_{e^{im\theta}f} = c_1\left(T_{e^{i\theta}\widetilde{\phi}}\right)^m, \qquad (2.4)$$

and

$$T_{e^{il\theta}g} = c_2\left(T_{e^{i\theta}\widetilde{\psi}}\right)^l, \qquad (2.5)$$

for some constants c_1 and c_2. To avoid the trivial case, which is $T_{e^{im\theta}f}$ and $T_{e^{il\theta}g}$ being the zero operator, we assume that c_1 and c_2 are nonzero constants. Thus Equation (3) can be written as

$$c_1\left[\left(T_{e^{i\theta}\widetilde{\phi}}\right)^m, T_{e^{is\theta}\psi}\right](z^k) = c_2\left[T_{e^{ip\theta}\phi}, \left(T_{e^{i\theta}\widetilde{\psi}}\right)^l\right](z^k) \qquad (2.6)$$

for all $k \geq 0$, where $[A, B] = AB - BA$ denotes the commutator of the operators A and B.

Remark 1. i) If $T_{e^{ip\theta}\phi}$ and $T_{e^{is\theta}\psi}$ commute with each other, then [9, Proposition 2 and Lemma 2] imply that $T_{e^{i\theta}\widetilde{\psi}} = cT_{e^{i\theta}\widetilde{\phi}}$ for some constant c. Moreover, [9, Corollary 1] implies that all four Toeplitz operators $T_{e^{ip\theta}\phi}$, $T_{e^{is\theta}\psi}$, $T_{e^{im\theta}f}$ and $T_{e^{il\theta}g}$ commute with each other, and therefore they are all of the form constant times power of a single Toeplitz operator $T_{e^{i\theta}\widetilde{\phi}}$. So without loss of generality, we assume $[T_{e^{ip\theta}\phi}, T_{e^{is\theta}\psi}] \neq 0$ from now on.

ii) The case $p = s$ (resp. $l = m$) has been extensively studied and totally solved. See [6, 7, 13].

iii) We shall show that for a certain class of Toeplitz operators $T_{e^{ip\theta}\phi}$, $T_{e^{is\theta}\psi}$ if $(H1)$ and $(H2)$ hold, then $m = p$, $l = s$, and hence $c_1 = c_2$. In other words $T_{e^{im\theta}f} + T_{e^{il\theta}g}$ is simply constant times $T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi}$. In fact, if $m = p$ (resp. $l = s$), then $(H2)$ implies $l = s$ (resp. $m = p$). Moreover, equations (2.4) and (2.5) imply $T_{e^{im\theta}f} = c_1T_{e^{ip\theta}\phi}$ and $T_{e^{il\theta}g} = c_2T_{e^{is\theta}\psi}$, for some constants c_1, c_2. Thus Equation (2.6) becomes

$$c_1\left[T_{e^{ip\theta}\phi}, T_{e^{is\theta}\psi}\right](z^k) = c_2\left[T_{e^{ip\theta}\phi}, T_{e^{is\theta}\psi}\right](z^k)$$

for all $k \geq 0$, and therefore $c_1 = c_2$ since we assume that

$$[T_{e^{ip\theta}\phi}, T_{e^{is\theta}\psi}] \neq 0.$$

Quasihomogeneous Toeplitz operators have the interesting property of acting on the elements of the orthogonal basis of L^2_a as shift operators with holomorphic weight [9]. In fact, if ϕ is a bounded radial function and p a positive integer, then for all $k \geq 0$ we have

$$\begin{aligned} &T_{e^{ip\theta}\phi}(\zeta^k)(z) \\ &= \int_0^1\int_0^{2\pi}\phi(r)r^k\sum_{j=0}^{\infty}(j+1)e^{i(k+p-j)\theta}r^jz^j\frac{1}{\pi}rdrd\theta \\ &= 2(k+p+1)\int_0^1\phi(r)r^{2k+p+1}dr\ z^{k+p}. \end{aligned}$$

Now, we define the Mellin transform of a function ϕ in $L^1([0,1], rdr)$ by

$$\mathcal{M}(\phi)(z) = \int_0^1\phi(r)r^{z-1}dr.$$

It is easy to see that $\mathcal{M}(\phi)$ is bounded and holomorphic in the right-half plane $\{z \in \mathbb{C} | \Re z > 2\}$. Thus

$$T_{e^{ip\theta}\phi}(\zeta^k)(z) = 2(k+p+1)\mathcal{M}(\phi)(2k+p+2)z^{k+p}.$$

The class of symbols we will be dealing with are those of the form $e^{ip\theta}\phi$ where $\phi(r) = r^{(2M+1)p}$, with $M \geq 1$ being integer. It has been shown in [7, Remark 15, ii)] that in this case the root $T_{e^{i\theta}\widetilde{\phi}}$ exists and $\widetilde{\phi}$ is a polynomial in r whose Mellin transform satisfies

$$\mathcal{M}\left(r\widetilde{\phi}\right)(z) = \frac{\prod_{j=0}^{M-1}(z+2jp+2p)}{\prod_{j=0}^{M}(z+2jp+2)}, \text{ for } \Re z > 2. \qquad (2.7)$$

We are now ready to state and prove our main result.

3 Main result

Theorem 3.1. *Let $\phi(r) = r^{(2M+1)p}$ and $\psi(r) = r^{(2N+1)s}$ with $p < s$, M, and N being all integers greater or equal to 1. If there exist $m, l \in \mathbb{N}$ and nontrivial radial functions f, g such that $(H1)$ and $(H2)$ are satisfied, then $m = p$, $l = s$ and*

$$T_{e^{im\theta}f} + T_{e^{il\theta}g} = c\left(T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi}\right)$$

for some constant c.

Proof. We shall keep the same notation introduced earlier that is $T_{e^{i\theta}\widetilde{\phi}}$ (resp. $T_{e^{i\theta}\widetilde{\psi}}$) is the root of $T_{e^{ip\theta}\phi}$ (resp. $T_{e^{is\theta}\psi}$). Since $(H1)$ and $(H2)$ are satisfied, equations (2.4), (2.5) and (2.6) hold. Moreover, using (2.7), Equation (6) becomes

$$c_1\left(R_1P_1 - R_2P_2\right) = c_2\left(R_3P_3 - R_4P_4\right), \text{ for all } k \geq 0 \tag{3.1}$$

where

$$\begin{aligned}
R_1 &= \frac{k+s+1}{k+(N+1)s+1},\\
P_1 &= \prod_{j=1}^{M}\left[\frac{k+s+jp+1}{k+s+m+jp+1}\right],\\
R_2 &= \frac{k+m+s+1}{k+m+(N+1)s+1},\\
P_2 &= \prod_{j=1}^{M}\left[\frac{k+jp+1}{k+m+jp+1}\right],\\
R_3 &= \frac{k+l+p+1}{k+l+(M+1)p+1},\\
P_3 &= \prod_{j=1}^{N}\left[\frac{k+js+1}{k+l+js+1}\right],\\
R_4 &= \frac{k+p+1}{k+(M+1)p+1},\\
P_4 &= \prod_{j=1}^{N}\left[\frac{k+p+js+1}{k+l+p+js+1}\right].
\end{aligned}$$

The proof is mainly to show that Equation (3.1) is true if and only if $m = p$ and so $l = s$ by $(H2)$. For the sufficiency, it is clear that if $m = p$ and so $l = s$, Equation (3.1) is reduced for all $k \geq 0$, to

$$\begin{aligned}
&c_1\Big[\frac{k+s+1}{k+(N+1)s+1}\frac{k+s+p+1}{k+s+(M+1)p+1} - \\
&\quad\frac{k+p+s+1}{k+p+(N+1)s+1}\frac{k+p+1}{k+(M+1)p+1}\Big] = \\
&c_2\Big[\frac{k+s+p+1}{k+s+(M+1)p+1}\frac{k+s+1}{k+(N+1)s+1} - \\
&\quad\frac{k+p+1}{k+(M+1)p+1}\frac{k+p+s+1}{k+p+(N+1)s+1}\Big].
\end{aligned}$$

Obviously this is possible if and only if $c_1 = c_2$, and therefore the desired result is obtained. To prove the necessity, we shall proceed by contradiction. We shall assume $m \neq p$, i.e., $l \neq s$ and we shall show that the set of poles of the left-hand side (LHS) of Equation (3.1) is not equal to the set of poles of its right-hand side (RHS). To do so, either we exhibit one pole in the LHS (resp. RHS) and show that it does not appear in the RHS (resp. LHS) or we count the poles of each side and show that the numbers of poles are distinct. For easiness, we will call pole any term in the denominator of R_i or P_i. For example, $k + s + m + Mp + 1$ is a pole obtained by taking $j = M$ in the denominator of P_1.

The key of this proof by contradiction is the following: first we shall assume $m \neq tp$ for $t \geq 1$, and second $m = tp$ but for $t \geq 2$. Thus, in the end and after reaching the contradiction, we shall be left with the only possibility $m = p$.

I. Assume $m \neq tp$ for $t \geq 1$

In this case none of the terms in the denominator of P_2 can be canceled by its numerator. Let us consider the pole $k+m+p+1$ obtained by taking $j = 1$ in the denominator P_2. Obviously, this pole is not eliminated by the numerator of R_2 because $s > p$. We shall show that this pole does not appear in RHS:

i) $k+m+p+1$ is not equal to the pole $k+l+(M+1)p+1$ of R_3. In fact, if this were not the case we would have $m = l + Mp$, which is impossible because $m < l$.

ii) $k+m+p+1$ is not equal to the pole $k+l+js+1$ of P_3 for any $1 \leq j \leq N$. In fact if this were not the case, then $m+p = l+js$, which is not possible because $m < l$ and $p < s$.

iii) $k + m + p + 1$ is not equal to the pole $k + (M + 1)p+1$ of R_4, because if this were not the case, we would have $m = Mp$, which would contradict our assumption that m is not a multiple of p.

iv) $k+m+p+1$ not equal to the pole $k+l+p+js+1$ of P_4 for any $1 \leq j \leq N$ since if this were not the case, we would have $m = l + js$, which is not possible because $m < l$.

We conclude by saying that, under the assumption $m \neq tp$ for $t \geq 1$, Equation (3.1) cannot be satisfied since we are able the find a pole from LHS that does not appear in RHS.

II. Assume $m = tp$ for $t \geq 2$

We shall discuss two situations: $M \leq t$ and $M > t$. In fact when $M > t$, terms of the numerator of P_1 (resp. P_2) would cancel some poles of the denominator. This is not the case when $M \leq t$.

• 1^{st} Situation: $M \leq t$. Consider the pole $k+m+p+1 = k + (t+1)p + 1$ of P_2 obtained by taking $j = 1$. Since $M \leq t$, this pole is not canceled by the numerator of P_2. Also, since $p < s$ the same pole is not canceled by the numerator of R_2. Moreover, it is easy to see that $k+m+p+1$ is equal to none of the poles of R_3, P_3 and P_4 because $m < l$ and $p < s$. We still have to check if $k+m+p+1$ can be equal to the pole $k+(M+1)p+1$ of R_4. In this case $m = Mp$, i.e., $t = M$. Now, consider the pole $k + s + m + p + 1$ of P_1 obtained by taking $j = 1$:

i) $k + s + m + p + 1$ is not equal to the pole $k + l + (M + 1)p + 1$ of R_3. In fact if this were not the case, we would have $Mp = p$, which is impossible because $M = t \geq 2$.

ii) $k+s+m+p+1$ is not equal to the pole $k+(M+1)p+1$ of R_4. In fact if this were not the case, we would have $s+m = Mp$, which is impossible because $m = Mp$ and $s > 0$.

iii) $k+s+m+p+1$ is not equal to the pole $k+l+p+js+1$ of P_4 for any $1 \leq j \leq N$ since if this were not the case, we would have $p = js$, which is impossible because $s > p$.

iv) $k+s+m+p+1$ is not equal to the pole $k+l+js+1$ of P_3 for any $1 \leq j \leq N$. In fact if this were not the case, we would have $2p = js$ for $1 \leq j \leq N$. Now since $s > p$, we must have $2p = s$, i.e., $j = 1$, and so $l = s+m-p = (M+1)p$. Then:

  a) If $M = 2$, the pole $k+m+p+1 = k+3p+1$ of P_2 obtained by taking $j = 1$ does not appear in RHS because the pole of R_4 is eliminated by the first term of the numerator of P_4 (since $s = 2p$) and this for any $N \geq 1$.

  b) If $M \geq 3$, the pole $k+Mp+3p+1$ appears at least twice in LHS when taking $j = 1$ in the denominator of P_1 and $j = 3$ in the denominator of P_2. However this same pole appears at most once in RHS when taking $j = 1$ in the denominator of P_3.

Therefore, $k+m+p+1$ cannot be equal to $k+(M+1)p+1$, and hence we conclude by saying that, under the assumption $m = tp$ for $t \geq 2$ and when $M \leq t$, Equation (3.1) is not satisfied since there exists a pole of LHS that is not a pole of RHS.

• 2^{nd} Situation: $M > t \geq 2$. In this case poles of P_1 (resp. P_2) are canceled by terms of its numerator, and Equation (3.1) becomes

$$c_1\Big[\frac{k+s+1}{k+(N+1)s+1}\frac{\prod_{j=1}^{t} k+s+jp+1}{\prod_{j=M-t+1}^{M} k+s+tp+jp+1} - \frac{k+tp+s+1}{k+tp+(N+1)s+1}\frac{\prod_{j=1}^{t} k+jp+1}{\prod_{j=M-t+1}^{M} k+tp+jp+1}\Big]$$
$$= c_2\Big[\frac{k+l+p+1}{k+l+(M+1)p+1}\prod_{j=1}^{N}\frac{k+js+1}{k+l+js+1} - \frac{k+p+1}{k+(M+1)p+1}\prod_{j=1}^{N}\frac{k+p+js+1}{k+l+p+js+1}\Big]. \quad (3.2)$$

At this stage, it is very important to notice that LHS has at most $2t+2$ poles and at least $2t$ poles. In fact, the pole of R_1 could be canceled by the numerator of P_1, and the numerator of R_2 could cancel a pole of P_2. However, RHS has at most $2N+2$ poles. Thus, if $N < t-1$ Equation (3.2) cannot be satisfied, and hence we must have $N \geq t-1$.

Now, consider the pole $k+s+tp+(M-t+1)p+1 = k+s+(M+1)p+1$ of P_1 obtained by taking $j = M-t+1$. Clearly it is not canceled by the numerator of R_1. We shall show that this pole does not appear in RHS:

i) $k+s+(M+1)p+1$ is not equal to the pole $k+l+(M+1)p+1$ of R_3. In fact if this were not the case, we would have $s = l$, and hence $m = p$, which contradicts our assumption that $m = tp$ for $t \geq 2$.

ii) Clearly $k+s+(M+1)p+1$ is not equal to the pole $k+(M+1)p+1$ of R_4.

iii) We will prove that $k+s+(M+1)p+1$ is not equal to the pole $k+l+js+1$ of P_3 for any $1 \leq j \leq N$. In fact if this were not the case, we would have $js = (M-t+2)p$. Since $s > p$, we must have $j < M-t+2$. We denote such j by j^*, i.e, $1 \leq j^* \leq N$, $j^* < M-t+2$ and $j^*s = (M-t+2)p$. We shall discuss the following cases:

(iii), 1^{st} Case:

If $M = 3$. Then $t = 2$, $M-t+2 = 3$ and so $j^* = 1$ or 2.

(a) If $j^* = 1$, then $s = 3p$ and $l = m+s-p = 4p$. Thus Equation (3.2) becomes

$$c_1\Big[\frac{k+3p+1}{k+3(N+1)p+1}\frac{(k+4p+1)(k+5p+1)}{(k+7p+1)(k+8p+1)} - \frac{1}{k+(3N+5)p+1}\frac{(k+p+1)(k+2p+1)}{k+4p+1}\Big]$$
$$= c_2\Big[\frac{k+5p+1}{k+8p+1}\prod_{j=1}^{N}\frac{k+3jp+1}{k+(4+3j)p+1} - \frac{k+p+1}{k+4p+1}\prod_{j=1}^{N}\frac{k+(1+3j)p+1}{k+(5+3j)p+1}\Big].$$

It is easy to see that the pole $k+4p+1$ of R_4 is canceled by the first term of the numerator of P_4. Hence this pole appears in LHS, as a pole of P_2, but does not appear in RHS. Therefore j^* cannot be 1.

(b) If $j^* = 2$, then $2s = 3p$, $l = p+s$, and Equation (3.2) becomes

$$c_1\Big[\frac{k+s+1}{k+(N+1)s+1}\frac{(k+s+p+1)(k+s+2p+1)}{(k+s+4p+1)(k+s+5p+1)} \frac{k+2p+s+1}{k+2p+(N+1)s+1}\frac{(k+p+1)(k+2p+1)}{(k+4p+1)(k+5p+1)}\Big]$$
$$= c_2\Big[\frac{k+2p+s+1}{k+s+5p+1}\prod_{j=1}^{N}\frac{k+js+1}{k+p+(j+1)s+1} - \frac{k+p+1}{k+4p+1}\prod_{j=1}^{N}\frac{k+p+js+1}{k+2p+(j+1)s+1}\Big].$$

We observe that when $N \geq 3$ the pole $k+s+5p+1$ appears only once in LHS from the denominator of P_1, however the same pole appears at least twice in RHS from the denominator of R_3 and from taking $j=2$ in the denominator of P_4 ($3s = s+2s = s+3p$). Note that this pole is not canceled by either the numerator of P_4 or R_4. It is also worth mentioning here that this pole is not canceled when we transpose P_3R_3 to LHS and reduce to a common denominator. Now, when $N=2$ (N cannot be 1 because $N \geq j^*$ and $j^*=2$), Equation (3.2) becomes

$$c_1\Big[\frac{k+s+1}{k+3s+1}\frac{(k+s+p+1)(k+s+2p+1)}{(k+s+4p+1)(k+s+5p+1)} - \frac{k+s+2p+1}{k+2p+3s+1}\frac{(k+p+1)(k+2p+1)}{(k+4p+1)(k+5p+1)}\Big]$$
$$=$$
$$c_2\Big[\frac{k+2p+s+1}{k+s+5p+1}\frac{(k+s+1)(k+2s+1)}{(k+p+2s+1)(k+p+3s+1)} - \frac{k+p+1}{k+4p+1}\frac{(k+p+s+1)(k+p+2s+1)}{(k+2p+2s+1)(k+2p+3s+1)}\Big].$$

By equating the poles in both sides, we must have $k+3s+1 = k+p+2s+1$ i.e., $s=p$ which is impossible because $s>p$.

We conclude by saying that, when $M=3$ and $j^*s = (M-t+2)p$, the pole $k+s+(M+1)p+1$ cannot be equal to the pole $k+l+js+1$ of P_3 for any $1 \leq j \leq N$.

(iii), 2^{nd} Case:

Assume $M \geq 4$. We shall discuss the cases $t=2$ and $t \geq 3$. In fact when $t \geq 3$, the denominator of P_2 contains at least three poles one of them is $k+m+(M-t+3)p+1 = k+m+j^*s+p+1$ obtained by taking $j = M-t+3$.

(a) Suppose $t \geq 3$. We show that the pole $k+m+j^*s+p+1$ of P_2 does not appear in RHS:

1. $k+m+j^*s+p+1$ is not equal to the pole $k+l+(M+1)p+1$ of R_3. In fact if this were not the case, we would have $(j^*-1)s = (M-1)p$. But since $j^*s = (M-t+2)p$, we would obtain that $s = (3-t)p$, which is impossible because $t \geq 3$.

2. $k+m+j^*s+p+1$ is not equal to the pole $k+l+js+1$ of P_3 for any $1 \leq j \leq N$. In fact if this were not the case, we would have $2p = (1+j-j^*)s$. This is possible only if $j=j^*$, i.e., $s=2p$, and Equation (3.2) becomes

$$c_1\Big[\frac{k+2p+1}{k+2(N+1)p+1}\frac{\prod_{j=1}^{t} k+(2+j)p+1}{\prod_{j=M-t+1}^{M} k+(2+t+j)p+1} - \frac{k+(t+2)p+1}{k+tp+2(N+1)p+1}\frac{\prod_{j=1}^{t} k+jp+1}{\prod_{j=M-t+1}^{M} k+(t+j)p+1}\Big]$$
$$=$$
$$c_2\Big[\frac{k+(t+2)p+1}{k+(t+M+2)p+1}\prod_{j=1}^{N}\frac{k+2jp+1}{k+tp+p+js+1} - \frac{k+p+1}{k+(M+1)p+1}\prod_{j=1}^{N}\frac{k+(1+2j)p+1}{k+(t+2)p+js+1}\Big].$$

Now observe that the pole $k+(M+3)p+1$ appears twice in LHS from $j = M-t+1$ in the denominator of P_1 and also from $j = M-t+3$ in the denominator of P_2. However this same pole appears only once in RHS when $j=j^*$ in the denominator of P_3. In fact $k+tp+p+j^*s+1 = k+tp+p+(M-t+2)p+1 = k+(M+3)p+1$. Moreover $k+(M+3)p+1$ cannot be equal to $k+(t+2)p+js+1$ for any $1 \leq j \leq N$ because if this were the case, and since $j^*s = (M-t+2)p$, we would have $(j^*-j)s = p$, which is impossible since $s>p$. It is worth mentioning here that this pole $k+(M+3)p+1$ is not canceled when we transpose P_2R_2 to the RHS and reduce to common denominator. Therefore $k+m+j^*s+p+1$ cannot be equal to $k+l+js+1$ for any $1 \leq j \leq N$.

3. Clearly $k+m+j^*s+p+1$ is not equal to $k+(M+1)p+1$ because $k+m+j^*s+p+1 = k+(M+3)p+1$.

4. $k+m+j^*s+p+1$ is not equal to $k+l+p+js+1$ for any $1 \leq j \leq N$ because if this were not the case, we would have $p = (1+j-j^*)s$, which is impossible since $s>p$.
We conclude by saying that, when $M \geq 4$, $t \geq 3$, and $j^*s = (M-t+2)p$, the pole $k+m+j^*s+p+1$ does not appear in RHS.

(b) Assume $t=2$. In this case LHS has at most six poles and at least five poles. In fact the pole of R_1 is canceled by the numerator of P_1 when $N=1$ and $s=2p$. On the other hand, RHS has at most $2N+2$ poles, and so Equation (3.2) cannot be satisfied if $N=1$. Moreover, since $m = tp = 2p$, we have that $l = p+s$ and thus none of the terms of the numerator of P_3 (resp. P_4) can cancel the poles of P_3 (resp. P_4). Hence if $N \geq 4$, Equation (3.2) is not satisfies because RHS has at least $2N \geq 8$ poles. Finally, we have to check the cases $N=2$ and $N=3$:

1. If $N=2$, then Equation (3.2) becomes

$$c_1\Big[\frac{k+s+1}{k+3s+1}\frac{(k+s+p+1)(k+s+2p+1)}{(k+s+(M+1)p+1)\,(k+s+(M+2)p+1)} -\frac{k+2p+s+1}{k+2p+3s+1}\frac{(k+p+1)(k+2p+1)}{(k+(M+1)p+1)\,(k+(M+2)p+1)}\Big]$$
$$=$$
$$c_2\Big[\frac{k+2p+s+1}{k+s+(M+2)p+1}\frac{(k+s+1)(k+2s+1)}{(k+p+2s+1)(k+p+3s+1)} -\frac{k+p+1}{k+(M+1)p+1}\frac{(k+p+s+1)(k+p+2s+1)}{(k+2p+2s+1)(k+2p+3s+1)}\Big].$$

Now, it is not hard to see that when trying to equate the poles from both sides we reach one of the following contradictions namely either "$2s=Mp$ and $2s=(M+1)p$" or "$s=p$".

2. If $N=3$, then Equation (3.2) becomes

$$c_1\Big[\Big(\frac{k+s+1}{k+4s+1}\Big)\Big(\frac{(k+s+p+1)(k+s+2p+1)}{(k+s+(M+1)p+1)\,(k+s+(M+2)p+1)}\Big) -\Big(\frac{k+2p+s+1}{k+2p+4s+1}\Big)\Big(\frac{(k+p+1)(k+2p+1)}{(k+(M+1)p+1)\,(k+(M+2)p+1)}\Big)\Big]$$
$$=$$
$$c_2\Big[\Big(\frac{k+2p+s+1}{k+s+(M+2)p+1}\Big)\Big(\frac{(k+s+1)(k+2s+1)(k+3s+1)}{(k+p+2s+1)(k+p+3s+1)(k+p+4s+1)}\Big) -\Big(\frac{k+p+1}{k+(M+1)p+1}\Big)\Big(\frac{(k+p+s+1)(k+p+2s+1)(k+p+3s+1)}{(k+2p+2s+1)(k+2p+3s+1)(k+2p+4s+1)}\Big)\Big].$$

This equality is possible only if both poles $k+s+(M+2)p+1$ and $k+(M+1)p+1$ in RHS are canceled so that the numbers of poles on both sides are equal. Notice that these are the only two poles that might be canceled in RHS. Thus we must have "$s=(M+2)p$ or $2s=(M+2)p$" and "$s=Mp$ or $2s=Mp$ or $3s=Mp$". The only possible combination is to have "$s=2p$ and $M=2$". But this cannot be true because $M\geq 4$.

Hence when $M\geq 4$, $t=2$, and $j^{*}s=(M-t+2)p$, Equation (3.2) cannot be satisfied.

Therefore, we conclude by saying that the pole $k+s+(M+1)p+1$ cannot be equal to $k+l+js+1$ for any $1\leq j\leq N$ in all possible situations.

iv) We will show that $k+s+(M+1)p+1$ is not equal to the pole $k+l+p+js+1$ of P_4 for any $1\leq j\leq N$. In fact if this were not the case, we would have $js=(M-t+1)p$ for some $1\leq j\leq N$. Let us denote such j by $\widetilde{j}$. Observe that $\widetilde{j}<(M-t+1)$ because $s>p$. We shall discuss the following cases:

(iv), 1^{st} Case:

If $M=3$. Then $t=2$, and so $\widetilde{j}=1$. Moreover we have $m=2p$, $s=2p$ and $l=3p$. Thus Equation (3.2) becomes

$$c_1\Big[\frac{k+2p+1}{k+2(N+1)p+1}\frac{(k+3p+1)(k+4p+1)}{(k+6p+1)(k+7p+1)} -\frac{1}{k+2(N+2)p+1}\frac{(k+p+1)(k+2p+1)}{k+5p+1}\Big]$$
$$=$$
$$c_2\Big[\frac{k+4p+1}{k+7p+1}\prod_{j=1}^{N}\frac{k+2jp+1}{k+(3+2j)p+1} -\frac{k+p+1}{k+4p+1}\prod_{j=1}^{N}\frac{k+(1+2j)p+1}{k+(4+2j)p+1}\Big].$$

It is easy to see that when $N=1$ the pole $k+4p+1$ of R_1 is canceled by the numerator of P_1, and so it does not appear in LHS. However, it appears in RHS as a pole of R_4. Now, when $N\geq 2$, LHS has 5 poles but RHS has $2N+2$ poles. Therefore, we conclude that when $M=3$, the pole $k+s+(M+1)p+1$ is not equal to $k+l+p+js+1$ for any $1\leq j\leq N$.

(iv), 2^{nd} Case:

Assume $M\geq 4$. We consider the pole $k+tp+(M-t+2)p+1=k+m+\widetilde{j}s+p+1$ of P_2 obtained by taking $j=M-t+2$. We shall show that this pole does not appear in RHS:

(a) $k+m+\widetilde{j}s+p+1$ is not equal to the pole $k+l+(M+1)p+1$ of R_3 because if this were not the case, we would have $(\widetilde{j}-1)s=(M-1)p$. But since $\widetilde{j}s=(M-t+1)p$, we must have $s=(2-t)p$, which is impossible because $t\geq 2$.

(b) Clearly $k+m+\widetilde{j}s+p+1$ is not equal to the pole $k+(M+1)p+1$ of R_4 because if this were not true, we would have $p=0$.

(c) $k+m+\widetilde{j}s+p+1$ is not equal to the pole $k+l+p+js+1$ of P_4 for any $1\leq j\leq N$ because if this were not the case, we would have $(j+1-\widetilde{j})s=p$, which is not possible because $s>p$.

(d) We show that $k+m+\widetilde{j}s+p+1$ is not equal to the pole $k+l+js+1$ of P_3 for any $1\leq j\leq N$. If this were not the case, we would have $(j+1-\widetilde{j})s=2p$. This is possible only when $j=\widetilde{j}$, i.e., $s=2p$. Here, we shall make the distinction between two cases t even and t odd:

1. If $t=2$ and $N=1$, then LHS has five poles and RHS has four poles. Hence, Equation (3.2) cannot be satisfied.

2. If t is even and $N \geq 2$, then we have the following cases: First, when "$M = 4$ and $t = 2$" (we recall that $t \leq M-1$ and t is even), LHS has exactly six poles, however RHS has either five (when $N = 2$) or seven (when $N = 3$) or $2N$ (when $N \geq 4$) poles. Therefore Equation (3.2) is not satisfied. Second, when "$M \geq 5$ and $M-2 \leq t \leq M-1$" the pole $k + (t+5)p + 1$ appears twice in LHS by taking $j = 3$ in the denominator of P_1 and $j = 5$ in the denominator of P_2, however the same pole appears only once in RHS by taking $j = 2$ in the denominator of P_3. Hence Equation (3.2) cannot be satisfied. Finally, when "$M \geq 5$ and $t \leq M-3$" the pole $k + l + s + 1 = k + (t+3)p + 1$ of P_3 (notice that this pole cannot be eliminated by either the numerator of R_3 or the numerator of P_3) obtained by taking $j = 1$ does not appear in LHS since $M - t + 1 \geq 4$. Thus Equation (3.2) is again not satisfied. We conclude by saying that when t is even, $k + m + \widetilde{j}s + p + 1$ cannot be equal to $k + l + js + 1$ for any $1 \leq j \leq N$.

3. If t is odd (and so $t \geq 3$ because we are assuming $m = tp$ for $t \geq 2$), then Equation (3.2) becomes

$$c_1 \left[\frac{k+2p+1}{k+2(N+1)p+1} \frac{\prod_{j=1}^{t} k+(2+j)p+1}{\prod_{j=M-t+1}^{M} k+(2+t+j)p+1} - \frac{k+(t+2)p+1}{k+tp+2(N+1)p+1} \frac{\prod_{j=1}^{t} k+jp+1}{\prod_{j=M-t+1}^{M} k+(t+j)p+1} \right]$$

$$=$$

$$c_2 \left[\frac{k+(t+2)p+1}{k+(t+M+2)p+1} \frac{\prod_{j=1}^{\frac{t+1}{2}} k+2jp+1}{\prod_{j=N-\frac{t+1}{2}+1}^{N} k+(t+1+2j)p+1} - \frac{k+p+1}{k+(M+1)p+1} \frac{\prod_{j=1}^{\frac{t+1}{2}} k+(1+2j)p+1}{\prod_{j=N-\frac{t+1}{2}+1}^{N} k+(t+2+2j)p+1} \right].$$

It is easy to see that LHS has at least $2t+1$ poles (in fact if $t = M-1$, then the first pole of P_2 obtained by taking $j = M-t+1 = 2$ is canceled by the numerator of R_2), however RHS has at most $t+3$ poles. But $2t+1 = t+3$ if and only if $t = 2$, which is impossible because we are assuming t is odd. Hence, when t is odd $k+m+\widetilde{j}s+p+1$ cannot be equal to $k + l + js + 1$ for any $1 \leq j \leq N$.

We conclude by saying that the pole $k+m+\widetilde{j}s+p+1$ does not appear in RHS.

This finishes proving that the pole $k+s+(M+1)p+1$ cannot be equal to $l + p + js + 1$ for any $1 \leq j \leq N$. Therefore the pole $k + s + (M+1)p + 1$ from LHS does not appear in RHS. We conclude by saying that, under the assumption $m = tp$ for $t \geq 2$, Equation (3.2) cannot be satisfied since we are always able the find a pole from LHS that does not appear in RHS. Therefore Equation (3.1) is true if and only if $m = p$, which implies $l = s$ by $(H2)$, and so $c_1 = c_2$ by Remark 1, iii). Finally, we obtain the desired result namely

$$T_{e^{im\theta}f} + T_{e^{il\theta}g} = c\left(T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi}\right)$$

□

Remark 2. If $m_1, l_1, m_2, l_2, f_1, g_1, f_2, g_2$ are as in Theorem 1, then we will have

$$T_{e^{im_1\theta}f_1} + T_{e^{il_1\theta}g_1} = c\left(T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi}\right),$$

and

$$T_{e^{im_2\theta}f_2} + T_{e^{il_2\theta}g_2} = c'\left(T_{e^{ip\theta}\phi} + T_{e^{is\theta}\psi}\right).$$

for some constants c, c', and therefore $T_{e^{im_1\theta}f_1} + T_{e^{il_1\theta}g_1}$ and $T_{e^{im_2\theta}f_2} + T_{e^{il_2\theta}g_2}$ commute with each other. Thus, Theorem 1 is a partial confirmation of the conjecture in [9, p. 263] which states that if two Toeplitz operators, defined on the Bergman space of the unit disk $\mathbb{D}$, commute with a third one, none of them being the identity or the zero operator, then they commute with each other.

Acknowledgments. The authors are grateful to the reviewer for his/her careful and meticulous reading of the paper. The reviews are detailed and helpful to finalize the manuscript. The authors would like to kindly acknowledge them.

References

[1] S. Axler, Toeplitz operators, *http://www.axler.net/ToeplitzOperators.pdf*

[2] S. Axler and Ž. Čučković, Commuting Toeplitz operators with harmonic symbols, *Integral Equations Operator Theory* 14 (1991), no. 1, 1-12.

[3] S. Axler, Ž. Čučković, and N. V. Rao, Commutants of analytic Toeplitz operators on the Bergman space, *Proc. Amer. Math. Soc.* 128 (2000), no. 7, 1951-1953.

[4] A. Brown abd P. Halmos, Algebraic properties of Toeplitz operators, *J. reine angew. Math.*, 213 (1964), 89-102.

[5] Ž. Čučković, Commutants of Toeplitz operators on the Bergman space, *Pacific J. Math.* 162 (1994), no. 2, 277-285.

[6] Ž. Čučković, N. V. Rao, Mellin transform, monomial symbols, and commuting Toeplitz operators. *J. Funct. Anal.* 154 (1998), no. 1, 195-214.

[7] I. Louhichi, Powers and roots of Toeplitz operators, *Proc. Amer. Math. Soc.* 135 (2007), no. 5, 1465-1475.

[8] I. Louhichi and Anders Olofson, Characterizations of Bergman space Toeplitz operators with harmonic symbols, *J. Reine Angew. Math.* 617 (2008), 1-26

[9] I. Louhichi and N. V. Rao, Bicommutants of Toeplitz operators, *Arch. Math.* (Basel) 91, (2008), no. 3, 256-264.

[10] I. Louhichi and N. V. Rao, Roots of Toeplitz operators on the Bergman space, *Pacific J. Math.* 252 (2011), no. 1, 127-144.

[11] I. Louhichi, N. V. Rao, A. Yousef, Two questions on the theory of Toeplitz operators on the Bergman space, *Complex Anal. Oper. Theory* 3 (2009), no. 4, 881-889.

[12] I. Louhichi, E. Strouse, L. Zakariasy, Products of Toeplitz operators on the Bergman space, *Integral Equations Operator Theory* 54 (2006), no. 4, 525-539.

[13] I. Louhichi, L. Zakariasy, On Toeplitz operators with quasihomogeneous symbols, *Arch. Math.* (Basel) 85 (2005), no. 3, 248-257.

[14] E. Strouse and L. Zakariasy, Quasihomogeneous Toeplitz operators, *http://www.academia.edu/22874908/*

[15] N. Vasilevski, Bergman space structure, commutative algebras of Toeplitz operators and hyperbolic geometry, *Integral Equations Operator Theory* 46 (2003), no. 2, 235-251.

Issam Louhichi and Khitam Aqel
Department of Mathematics & Statistics,
College of Arts & Sciences,
American University of Sharjah,
P.O.Box 2666, Sharjah, UAE.
E-mail address: g00047623@aus.edu
E-mail address: ilouhichi@aus.edu