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Miyachi's Theorem for oscillator Lie groups

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Abstract

We formulate and prove an analogue of Miyachi's theorem for the oscillator groups, which is a semi-direct product of $\mathbb{R}$ with the Heisenberg group $\mathbb{H}_3$. This allows us to generalize and prove an analogue of the Hardy and Cowling-Price uncertainty principle on G.

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1 Introduction

The uncertainty principle in quantum mechanics states in rough terms that one cannot assign exact simultaneous values to the position and momentum of a physical system. Rather, these quantities can only be determined with some characteristic "uncertainties" that cannot become arbitrarily small simultaneously. This principle discovered by German theoretical physicist Werner Heisenberg in 1927, has deep implications on how we understand the universe and is behind various new technologies of the 21st century. It also has philosophical ramifications.

The uncertainty principle in harmonic analysis is a class of theorems which state that a nontrivial function and its Fourier transform can not both be too sharply localized. That is, it is impossible for a nonzero function and its Fourier transform to be simultaneously small. Here a concept of the smallness had taken different interpretations in different contexts.

For the classical Fourier transform $\mathcal{F}$ on $\mathbb{R}$, Hardy in [9] was first to assert that f and its Fourier transform $\widehat{f} = \mathcal{F}(f)$ cannot both be very small. More precisely, let a and b be positive constants and assume that f is a measurable function on $\mathbb{R}$ such that $|f(x)| \le Ce^{-a\pi x^2}$ and $|\widehat{f}(y)| \le Ce^{-b\pi y^2}$ for some positive constant C. Then $f = 0$ almost everywhere on $\mathbb{R}$ if $ab > 1$, f is a constant multiple of $e^{-a\pi x^2}$ if $ab = 1$, and there are infinitely many nonzero functions satisfying the assumptions if $ab < 1$.

Considerable attention has been devoted to giving generalizations of Hardy's theorm to other contexts. Most notably in 1983, Cowling and Price [8] gave an L^p version of Hardy's theorem which states that for $1 \le p, q \le +\infty$, at least one of them is finite, if $\|e^{a\pi x^2} f\|_p < \infty$ and $\|e^{a\pi y^2}\widehat{f}\|_q < \infty$, then $f = 0$ almost everywhere on $\mathbb{R}$ if $ab \ge 1$. Another generalization of Hardy's theorem given in 1997 by Miyachi [14] states that, if f is a measurable function on $\mathbb{R}$ such that $e^{a\pi|.|^2} f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, $e^{a\pi|t|^2} f(t) = u_1(t) + u_\infty(t)$ with $u_1 \in L^1(\mathbb{R})$ and $u_\infty \in L^\infty(\mathbb{R})$ and if

$$\int_{\mathbb{R}} \log^+ \left(\frac{e^{b\pi|\xi|^2} |\widehat{f}(\xi)|}{c} \right) \, d\xi < +\infty,$$

for some positive constants a, b and c, and $ab = 1$, then $f(x) = A\ e^{-a\pi x^2}$, $x \in \mathbb{R}$ and A is a positive constant. Here

$$\log^+(t) = \begin{cases} \log(t), & \text{if } t > 1 \\ 0, & \text{otherwise} \end{cases}$$

The following corollary is an immediate consequence of Miyachi's Theorem.

Corollary 1.1. *(cf. [1]) Let f be an integrable function on $\mathbb{R}$ and $1 \le p, q \le +\infty$ such that $e^{a\pi|.|^2} f \in L^p(\mathbb{R}) + L^q(\mathbb{R})$, for some positive a. Further assume that*

$$\int_{\mathbb{R}} e^{2b\pi\xi^2} |\widehat{f}(\xi)|^2 \log^+ \left(\frac{e^{b\pi|\xi|^2} |\widehat{f}(\xi)|}{c} \right) \, d\xi < +\infty,$$

for some positive numbers b and c. If $ab = 1$, then $f(x) = A\ e^{-a\pi x^2}$, $x \in \mathbb{R}$ and A is a positive constant. If $ab < 1$, then there are infinitely many linearly independent functions meeting the hypotheses. Otherwise $(ab > 1)$, f vanishes almost everywhere on $\mathbb{R}$.

This note aims to prove an analogue of Miyachi's Theorem in the context of oscillator Lie groups (Theorem 2.1). As a consequence of this theorem, we extend Hardy's and Cowling-Price's Theorems in the context of oscillator groups as well (Theorem 3.1 and Theorem 3.3 respectively). For a more general context one can consult [2].

1.1 Oscillator groups

The oscillator groups, called Warped Heisenberg Lie groups in [19], have interesting applications in conformal field theory in WZW models [15] and supergravity. These groups belong to a class of 3-step solvable connected Lie groups which are non-exponential. We recall the various definitions. An exponential Lie group is a real finite-dimensional Lie group G for which the exponential map $\mathfrak{g} \longrightarrow G$, from the Lie algebra of G to the group is a diffeomorphism. On the other hand, the derived series of $\mathfrak{g}$ is the series with terms $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}^{n-1}]$, for $n \geq 2$. We say $\mathfrak{g}$ is $k-$step solvable if $\mathfrak{g}^k = 0$. A solvable Lie group is a Lie group G which whose Lie algebra $\mathfrak{g}$ is a solvable Lie algebra.

Recall that the Heisenberg Lie algebra $\mathfrak{h}_3$ is a $2-$step real nilpotent Lie algebra of dimension 3 admitting a basis Z, X, Y such that the non vanishing brackets are $[X, Y] = Z$. Let $\mathbb{H}_3$ be the simply connected Lie group associated to the $\mathfrak{h}_3$. We identify $\mathbb{H}_3$ to the affine space $\mathbb{R}^3 = (\mathbb{R}^2) \times \mathbb{R}$ and any element x of $\mathbb{H}_3$ can be written as a column vector in $\mathbb{R}^3$,

$$x = ((\alpha, \beta), z)^t .$$

Let $r(t)$ be the rotation transformation

$$r(t) = \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix}, \ t \in \mathbb{R}.$$

We form the semi-direct product $G = \mathbb{H}_3 \rtimes \mathbb{R}$, where $\theta \in \mathbb{R}$ acts on $x \in \mathbb{H}_3$ by:

$$\theta \cdot x = (r(\theta)(\alpha, \beta), z)^t .$$

The associated oscillator algebra $\mathfrak{g}$ is defined as the direct sum of $\mathfrak{h}_3$ and a $1-$dimensional abelian Lie algebra A with the non-trivial brackets

$$[A, X] = -Y, \ [X, Y] = Z \text{ and } [A, Y] = X.$$

This group G is a connected and simply connected type I Lie group. We recall that a Lie group is type I if any coadjoint orbit is an open subset in its closure and any ℓ in $\mathfrak{g}^*$ is integral.

1.2 The Plancherel formula for oscillator groups

Let $\mathbb{H} = \mathbb{H}_3$. Since $\mathbb{Z} \subset \mathbb{R}$ acts trivially on $\mathbb{H}$, the center $Z(G)$ of G is equal to the direct product $Z(\mathbb{H}) \times \mathbb{Z}$ and $N = \mathbb{H} \rtimes \mathbb{Z}$ is a closed normal subgroup of G such that $G/N = \mathbb{T}$. Every element of G can be written as a triple (h, n, z) where $h \in \mathbb{H}, n \in \mathbb{Z}$ and $z \in \mathbb{T}$. We identify $\{(t, 0, 0) : \ t \in \mathbb{R}\}$, the center $Z(\mathbb{H})$ of $\mathbb{H}$, with $\mathbb{R}$. The elements of the unitary dual $\widehat{G}$ of the group G are described in [12], we just describe here those which intervene in the Plancherel formula. For $\lambda \in \mathbb{R}^\star$, let π_λ denote the unique element of $\widehat{\mathbb{H}}$ such that $\pi_\lambda|_{\mathbb{R}}$ is a multiple of the character $t \rightarrow e^{i\lambda t}$. Moreover, for $z \in \mathbb{T}$, let χ_z denote the associated character of $\mathbb{Z}$. Then, with $\pi_{\lambda,z} = \pi_\lambda \times \chi_z$,

$$\widehat{N} = \left(\widehat{\mathbb{H}/Z(\mathbb{H})} \times \widehat{\mathbb{Z}}\right) \cup \{\pi_{\lambda,z} : z \in \mathbb{T}, \lambda \in \mathbb{R}, \lambda \neq 0\} .$$

Since N is nilpotent and G/N is compact, G is unimodular. Furthermore, $\widehat{N}$ is a T_1-space and, since G/N is compact, this implies that G is type I. Actually, $\widehat{G}$ is even a T_1-space. Therefore we can apply the explicit Plancherel formula for compact group extensions established in Theorem 4.4 of [11]. Observe next that $\widehat{N/Z(\mathbb{H})}$ and $\widehat{G/Z(\mathbb{H})}$ are zero sets for the Plancherel measures of N and G, respectively the stability subgroup of every $\pi_{\lambda,z}$ is equal to G. Therefore, by Mackey's theory [13], $\pi_{\lambda,z}$ extends to an irreducible multiplier representation $\sigma_{\lambda,z}$ of G. However, since $G/N = \mathbb{T}$ and every multiplier on $\mathbb{T}$ is trivial [13], we can assume that $\sigma_{\lambda,z}$ is an ordinary representation of G. Then the elements of $\widehat{G}$ extending $\pi_{\lambda,z}$ are precisely the representations of the form $\sigma_{\lambda,z} \otimes \chi$, where χ is an arbitrary character of $\mathbb{T}$. For $n \in \mathbb{Z}$, let χ_n denote the associated character of $\mathbb{T}$. Then

$$\widehat{G} \backslash \widehat{G/Z(\mathbb{H})} = \{\sigma_{\lambda,z} \otimes \chi_n : \lambda \in \mathbb{R}, \lambda \neq 0, z \in \mathbb{T}, n \in \mathbb{Z}\} .$$

As mentioned above,

$$\widehat{N} \backslash \widehat{N/Z(\mathbb{H})} = \{\pi_{\lambda,z} : \lambda \in \mathbb{R}, \lambda \neq 0, z \in \mathbb{T}\} ,$$

and G acts trivially on $\widehat{N} \backslash \widehat{N/Z(\mathbb{H})}$.

Theorem 1.3 next concerning the Plancherel formula for the group G is proved in [11]. First, we define $\widehat{f}^{i,j}$ the partial Fourier transform with respect to the i^{th} and j^{th} variables. In the case of the oscillator groups, we compute explicitly the Hilbert-Schmidt norm of $(\sigma_{\lambda,z} \otimes \chi_n)(f)$.

Lemma 1.2. *We have*

$$\|(\sigma_{\lambda,z} \otimes \chi_n)(f)\|^2_{H.S} = \frac{1}{|\lambda|} \left(\int_{\mathbb{R}^2} |\widehat{\varphi}^h(x, y, \lambda, z, n)|^2 dxdy \right)$$

where $\varphi(., z, n) = \widehat{f}^{4,5}(., z, n)$ *and* $\widehat{\varphi}^h = \widehat{\varphi}^{1,2,3}$.

Proof. Let $f \in \mathcal{C}_c^\infty(G)$ and $h = (x, y, t)$ with $x, y \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$\begin{aligned} &(\sigma_{\lambda,z} \otimes \chi_n)(f) \\ &= \int_{\mathbb{H}} \int_{\mathbb{T}} \sum_{m \in \mathbb{Z}} \pi_\lambda(h) \chi_z(m) \chi_n(u) f(h, m, u) du dh \\ &= \int_{\mathbb{T}} \sum_{m \in \mathbb{Z}} e^{-imz} \pi_\lambda(f(., m, u)) e^{-inu} du \\ &= \pi_\lambda(\varphi(., z, n)) \end{aligned}$$

where $\varphi(., z, n) = \widehat{f}^{4,5}(., z, n)$.

Using [18], we get

$$\begin{aligned}
&\|(\sigma_{\lambda,z}\otimes\chi_n)(f)\|^2_{H.S}\\
&=\|\pi_\lambda(\varphi(.,z,n))\|^2_{H.S}\\
&=\int_{\mathbb{R}^2}|K_{\varphi(.,z,n)}(x,y)|^2dxdy\\
&=\int_{\mathbb{R}^2}|\widehat{\varphi}^{2,3}(y-x,-\lambda x,\lambda,z,n)|^2dxdy\\
&=\int_{\mathbb{R}^2}|\widehat{\varphi}^{2,3}(x,y,\lambda,z,n)|^2dx\frac{dy}{|\lambda|}.
\end{aligned}$$

We therefore obtain that:

$$\begin{aligned}
&\|(\sigma_{\lambda,z}\otimes\chi_n)(f)\|_{H.S}\\
&=\frac{1}{|\lambda|^{\frac12}}\left(\int_{\mathbb{R}^2}|\widehat{\varphi}^{2,3}(x,y,\lambda,z,n)|^2dxdy\right)^{\frac12}\\
&\text{(using the Plancherel formula)}\\
&=\frac{1}{|\lambda|^{\frac12}}\left(\int_{\mathbb{R}^2}|\widehat{\varphi}^{h}(x,y,\lambda,z,n)|^2dxdy\right)^{\frac12}\\
&=\frac{1}{|\lambda|^{\frac12}}\left(\int_{\mathbb{R}^2}|\widehat{\varphi}^{h}(x,y,\lambda,z,n)|^2dxdy\right)^{\frac12}\\
&=\int_{\mathbb{R}^2}|\widehat{\varphi}^{h}(x,y,\lambda,z,n)|^2dx\frac{dy}{|\lambda|}.
\end{aligned}$$

□

Theorem 1.3. *Let G be the oscillator groups. For $f\in L^1(G)\cap L^2(G)$, the Plancherel formula is given by:*

$$\|f\|_2^2=\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}^*}\int_{\mathbb{T}}\int_{\mathbb{R}^2}|\widehat{\varphi}^{2,3}(x,y,\lambda,z,n)|^2dxdydzd\lambda,$$

where $\varphi(x,y,\lambda,z,n)=\widehat{f}^{4,5}(x,y,\lambda,z,n)$.

1.3 Some useful inequalities

1.3.1 Minkowski's generalized inequality

Let (X,μ) and (Y,ν) be σ-finite measure spaces and F a measurable function on $X\times Y$. Then for any $r\geq 1$, we have

$$\left(\int_X\left(\int_Y|F(x,y)|\,dy\right)^r dx\right)^{1/r}\leq\int_Y\left(\int_X|F(x,y)|^r\,dx\right)^{1/r}dy.$$

1.3.2 Jensen's inequalities

- Let $(\Omega,\ \mu)$ be a probability measure space. If g is a real-valued μ-integrable function, and if φ is a convex function on the real line, then:

$$\varphi\left(\int_\Omega gd\mu\right)\leq\int_\Omega\varphi\circ gd\mu. \tag{1.1}$$

- Let C be a convex set in real vector space. Then for any sequence $(x_n)_{n\in\mathbb{Z}}$ from C and $\lambda_i\geq 0$ with $\sum_{n\in\mathbb{Z}}\lambda_n=1$. Then inequality (cf. [16])

$$\varphi\left(\sum_{n\in\mathbb{Z}}\lambda_n x_n\right)\leq\sum_{n\in\mathbb{Z}}\lambda_n\varphi(x_n). \tag{1.2}$$

holds for every convex function $\varphi: C\longrightarrow\mathbb{R}$.

2 Miyachi's Theorem for oscillator groups

We are now ready to prove an L^p-L^q analogue of Miyachi's Theorem for oscillator groups.

Let G be an oscillator group, and let

$$\Sigma=\{((0,x,y),\theta): x,y,\theta\in\mathbb{R}\} \tag{2.1}$$

which is a natural cross-section for the cosets of $\mathbb{R}=Z(\mathbb{H})$ in G. We normalize Haar measures on G and on $G/\mathbb{R}$ so that Weil's formula [3] holds for G, $\mathbb{R}$ and $G/\mathbb{R}$ and endow Σ with the image of the Haar measure on $G/\mathbb{R}$ under the homeomorphism

$$\mathbb{R}\,((0,x,y),\theta)\to((0,x,y),\theta)$$

between $G/\mathbb{R}$ and Σ.

Theorem 2.1. *Let G be an oscillator group, $a,\ b\in\mathbb{R}_+^*$ and $2\leq p,q\leq+\infty$. Let $f:G\to\mathbb{C}$ be a measurable function on G satisfying the following decay conditions:*

(i) $e^{a\pi\|(t,s)\|^2}|f(t,s)|\in L^p(G)+L^q(G)$, for all $t\in\mathbb{R}$ and $s\in\Sigma$, the section defined in (2.1).

(ii) $$\int_{\mathbb{R}}\int_{\mathbb{T}}\sum_{n\in\mathbb{Z}}e^{2b\pi(\lambda^2+|n|^2)}\int_{\mathbb{R}^2}|\widehat{\varphi}^h(x,y,\lambda,z,n)|^2dxdy\times\log^+\left(\frac{e^{b\pi(\lambda^2+|n|^2)}\left(\int_{\mathbb{R}^2}|\widehat{\varphi}^h(x,y,\lambda,z,n)|^2dxdy\right)^{\frac12}}{C}\right)dzd\lambda<+\infty$$

for some positive constant C. If $ab>1$, then $f=0$ almost everywhere on G. If $ab=1$, then $f(t,s)=e^{-a\pi t^2}f(0,s)$ for all $t\in\mathbb{R}$ and $s\in\Sigma$. When $ab<1$, there are infinitely many linearly independent functions satisfying (i) and (ii).

Proof. For $s\in\Sigma$, let f_s denote the function defined by $f_s(t)=f(t,s)$, let $g:\mathbb{R}\to\mathbb{C}$ be a measurable function defined by:

$$g(x)=\int_\Sigma(f_s*f_s^\star)(x)\ ds\quad x\in\mathbb{R}.$$

For a fixed Schwartz function φ on Σ, define the function F on $\mathbb{R}$ by

$$F(t)=\int_\Sigma f(t,s)\varphi(s)\ ds,\ t\in\mathbb{R}.$$

Our proof is decomposed into a series of three lemmas given next.

Lemma 2.2. *Let f be a measurable function on G satisfying the first condition (i) of Theorem 2.1, then*

$$e^{a\pi|\cdot|^2}F \in L^p(\mathbb{R}) + L^q(\mathbb{R}).$$

Proof. Starting from the first decay condition of the theorem 2.1 concerning f, we observe that

$$F(t) = \int_\Sigma e^{-a\pi(t^2+\|s\|^2)}\big(u_p(t,s)+u_q(t,s)\big)\varphi(s)ds$$

for all $t \in \mathbb{R}$. Therefore,

$$\begin{aligned} e^{a\pi t^2}F(t) &= \int_\Sigma e^{-a\pi\|s\|^2}u_p(t,s)\varphi(s)ds \\ &\quad + \int_\Sigma e^{-a\pi\|s\|^2}u_q(t,s)\varphi(s)ds. \end{aligned}$$

Let

$$F_j(t) = \int_\Sigma e^{-a\pi\|s\|^2}u_j(t,s)\varphi(s)ds, \quad j \in \{p,q\}.$$

Then $F_p \in L^p(\mathbb{R})$ and $F_q \in L^q(\mathbb{R})$. Indeed,

$$\begin{aligned} &\int_\mathbb{R} |\int_\Sigma e^{-a\pi\|s\|^2}u_p(t,s)\varphi(s)ds|^p dt \\ &\leq \int_\mathbb{R}(\int_\Sigma e^{-a\pi\|s\|^2}|u_p(t,s)||\varphi(s)|ds)^p dt \\ &\leq M_1\int_\mathbb{R}(\int_\Sigma e^{-a\pi\|s\|^2}|u_p(t,s)|ds)^p dt \\ &\text{(using Minkowski's inequality)} \\ &< M_1\Big(\int_\Sigma(\int_\mathbb{R} e^{-ap\pi\|s\|^2}|u_p(t,s)|^p dt)^{\frac{1}{p}}ds\Big)^p \\ &\text{(using Hölder's inequality)} \\ &< M_2\int_{\mathbb{R}\times\Sigma}|u_p(t,s)|^p dtds \\ &< +\infty, \end{aligned}$$

for some positive constants M_1 and M_2. Similarly, the same conclusion holds for F_q. □

We now prove the following lemma:

Lemma 2.3. *We have*

$$\widehat{g}(\lambda) = \sum_{n\in\mathbb{Z}}\int_\mathbb{T}\int_{\mathbb{R}^2}|\widehat{\varphi}^h(x,y,\lambda,z,n)|^2 dxdydz.$$

Proof. For $u \in L^1(\mathbb{R})$, define $u * f$ on G by $u*f(x) = \displaystyle\int_\mathbb{R} u(t)f(t^{-1}x)\,dt$ and then $g_u : \mathbb{R} \to \mathbb{C}$ by

$$g_u(t) = \int_\Sigma (u*f)_s * (u*f)_s^\star(t)ds.$$

Thus g_u is obtained by replacing f with $u * f$ in the definition of the function g. Then it is easily verified that

$$g_u(t) = \int_\Sigma (u*f_s)*(u*f_s)^\star(t)\;ds.$$

Thus, for every $\lambda \in \mathbb{R}$,

$$\begin{aligned} \widehat{g_u}(\lambda) = \int_\Sigma \left|\widehat{(u*f)}_s(\lambda)\right|^2 ds &= |\widehat{u}(\lambda)|^2\int_\Sigma\left|\widehat{f_s}(\lambda)\right|^2\;ds \\ &= |\widehat{u}(\lambda)|^2\,\widehat{g}(\lambda). \end{aligned}$$

Now by the inversion formula for $\mathbb{R}$

$$\begin{aligned} \int_\mathbb{R}\widehat{g_u}(\lambda)\;d\lambda &= g_u(0) \qquad (2.2)\\ &= \int_\Sigma\int_\mathbb{R}|u*f_s(t)|^2 dtds \\ &= \|u*f\|_2^2. \end{aligned}$$

Now fix λ and let $V_k(\lambda) = [\lambda - \frac{1}{2k}, \lambda + \frac{1}{2k}]$, for a positive integer k. Let $u_{k,m} \in L^1(\mathbb{R})$ be such that:

(i) $0 \leq \widehat{u_{k,m}} \leq 1$.

(ii) $(\widehat{u_{k,m}})_m$ converges pointwise to the characteristic function of $V_k(\lambda)$ when m goes to infinity.

Then using equation (2.2) and the Plancherel formula, we get:

$$\begin{aligned} &\widehat{g}(\lambda) \\ &= \lim_{k\to+\infty} k\int_{V_k(\lambda)}\widehat{g}(\gamma)d\gamma \\ &= \lim_{k\to+\infty} k\int_\mathbb{R}\lim_{m\to+\infty}\widehat{g_{u_{k,m}}}(\gamma)d\gamma \\ &= \lim_{k,m\to+\infty} k\int_\mathbb{R}\widehat{g_{u_{k,m}}}(\gamma)d\gamma \\ &= \lim_{k,m\to+\infty} k\;\|u_{k,m}*f\|_2^2 \\ &= \lim_{k\to+\infty} k\sum_{n\in\mathbb{Z}}\int_{V_k(\lambda)}\int_\mathbb{T}\|(\sigma_{\gamma,z}\otimes\chi_n)(f)\|_{\text{HS}}^2 dzd\gamma. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} &(\sigma_{\gamma,z}\otimes\chi_n)(f) \\ &= \int_G f(x)(\sigma_{\gamma,z}\otimes\chi_n)(x)dx \\ &= \int_{\Sigma\times\mathbb{R}} f(t,s)(\sigma_{\gamma,z}\otimes\chi_n)(t,s)dtds \\ &= \int_{\Sigma\times\mathbb{R}} f(t,s)(\sigma_{\gamma,z}\otimes\chi_n)(s)e^{-2\pi it\gamma}dtds \\ &= (\widetilde{\sigma_{\gamma,z}\otimes\chi_n})(f^\gamma), \end{aligned}$$

where $f^\gamma = \widehat{f}^{,1}(\gamma,.)$ is the Fourier transform with respect to the first variable. Therefore,

$$\widehat{g}(\lambda) = \lim_{k\to+\infty} k\sum_{n\in\mathbb{Z}}\int_{V_k(\lambda)}\int_\mathbb{T}\|(\widetilde{\sigma_{\gamma,z}\otimes\chi_n})(f^\gamma)\|_{\text{HS}}^2 dzd\gamma.$$

Now,

$$\begin{aligned}&\int_{\mathbb{R}} \|f^\gamma\|^2_{L^2(G/\mathbb{R}Z)} d\gamma\\ &= \int_{\mathbb{R}\times\Sigma} |f(t,s)|^2 dt ds\\ &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^*}\int_{\mathbb{T}} \|(\sigma_{\lambda,z}\otimes\chi_n)(f)\|^2_{\mathrm{HS}} dz d\lambda\\ &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}}\int_{\mathbb{T}} \|(\widetilde{\sigma_{\gamma,z}\otimes}\chi_n)(f^\gamma)\|^2_{\mathrm{HS}} dz d\gamma.\end{aligned}$$

If we replace f^γ by $(u*f)^\gamma$ for $u\in\mathcal{S}(\mathbb{R})$, the Schwartz space of $\mathbb{R}$, we get:

$$\begin{aligned}&\int_{\mathbb{R}} \|(u*f)^\gamma\|^2_{L^2(G/\mathbb{R}Z)} d\gamma\\ &\quad= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}}\int_{\mathbb{T}} \|(\widetilde{\sigma_{\gamma,z}\otimes}\chi_n)\left((u*f)^\gamma\right)\|^2_{\mathrm{HS}} dz d\gamma\\ &\int_{\mathbb{R}} |\widehat{u(\gamma)}|^2 \|f^\gamma\|^2_{L^2(G/\mathbb{R}Z)} d\gamma\\ &\quad= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}}\int_{\mathbb{T}} |\widehat{u(\gamma)}|^2 \|(\widetilde{\sigma_{\gamma,z}\otimes}\chi_n)(f^\gamma))\|^2_{\mathrm{HS}} dz d\gamma.\end{aligned}$$

This means that

$$\int_{\mathbb{T}} \|(\widetilde{\sigma_{\gamma,z}\otimes}\chi_n)(f^\gamma)\|^2_{\mathrm{HS}} dz = \|f^\gamma\|^2_{L^2(G/\mathbb{R}Z)}$$

We get therefore for $\lambda\in\mathbb{R}$:

$$\begin{aligned}\widehat{g}(\lambda) &= \lim_{k\longrightarrow+\infty} k\int_{V_k(\lambda)} \|f^\gamma\|_2^2 d\gamma\\ &= \lim_{k\longrightarrow+\infty} k\int_{V_k(\lambda)}\int_{\Sigma} |f^\gamma(s)|^2 ds d\gamma\\ &= \lim_{k\longrightarrow+\infty} \int_{\Sigma} k\int_{V_k(\lambda)} |f^\gamma(s)|^2 d\gamma ds.\end{aligned}$$

Furthermore, $k\displaystyle\int_{V_k(\lambda)} |f^\gamma(s)|^2 d\gamma = |f^{c_k}(s)|^2$ for some $c_k\in V_k(\lambda)$ and $|f^{c_k}(s)|^2 \le \left(\displaystyle\int_{\mathbb{R}} |f(t,s)|dt\right)^2$. If we can show that the integral

$$E = \int_{\Sigma}\left(\int_{\mathbb{R}} |f(t,s)|dt\right)^2 ds$$

converges, then we can apply the dominated convergence theorem. Indeed, by using Hölder's inequality, we get

$$\begin{aligned}E &\le \left(\int_{\mathbb{R}}\left(\int_{\Sigma} |f(t,s)|^2 ds\right)^{1/2} dt\right)^2\\ &\le \left(\int_{\mathbb{R}}\left(\int_{\Sigma} e^{-2a\pi\|(t,s)\|^2}(|u_p(t,s)|+|u_q(t,s)|)^2 ds\right)^{1/2} dt\right)^2\\ &\le \left(\int_{\mathbb{R}}\left(\int_{\Sigma} 2e^{-2a\pi\|(t,s)\|^2}(|u_p(t,s)|^2+|u_q(t,s)|^2) ds\right)^{1/2} dt\right)^2\\ &\le \left(\int_{\mathbb{R}} e^{-a\pi t^2}\left(\int_{\Sigma} 2e^{-a\pi\|(t,s)\|^2}(|u_p(t,s)|^2+|u_q(t,s)|^2) ds\right)^{1/2} dt\right)^2\\ &\le M\int_{\mathbb{R}\times\Sigma}\left(e^{-a\pi\|(t,s)\|^2}|u_p(t,s)|^2 + e^{-a\pi\|(t,s)\|^2}|u_q(t,s)|^2\right) ds dt.\end{aligned}$$

Let $p'=\frac{p}{2}\ge 1$ and $q'=\frac{q}{2}\ge 1$, we get using again Hölder's inequality:

$$\begin{aligned}&E\\ &\le M'\left(\left(\int_{\mathbb{R}\times\Sigma} |u_p(t,s)|^{2p'} ds dt\right)^{\frac{1}{p'}} + \left(\int_{\mathbb{R}\times\Sigma} |u_q(t,s)|^{2q'} ds dt\right)^{\frac{1}{q'}}\right)\\ &< +\infty,\end{aligned}$$

where M and M' are some positive constants. We finally get that:

$$\begin{aligned}\widehat{g}(\lambda) &= \int_{\Sigma} \lim_{k\to+\infty} |f^{c_k}(s)|^2 ds\\ &= \int_{\Sigma} |f^\lambda(s)|^2 ds\\ &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{T}}\int_{\mathbb{R}^2} |\widehat{\varphi}^h(x,y,\lambda,z,n)|^2 dx dy dz.\end{aligned}$$

This completes the proof of the lemma. □

We now investigate the second condition decay of the function F.

Lemma 2.4. *Let f and C be as in Theorem 2.1, then*

$$\int_{\mathbb{R}} e^{2b\pi\lambda^2}|\widehat{F}(\lambda)|^2 \log^+\left(\frac{e^{b\pi\lambda^2}|\widehat{F}(\lambda)|}{C'}\right) d\lambda < +\infty$$

for $C' = C\sqrt{\|\varphi\|_2^2 C_1}$ with φ and C_1 are given in the proof.

Proof. We have

$$\widehat{F}(\lambda) = \int_{\Sigma} \widehat{f}_s(\lambda)\varphi(s) ds,\ \lambda\in\mathbb{R}$$

and therefore

$$|\widehat{F}(\lambda)|^2 \le \|\varphi\|_2^2 \int_{\Sigma} |\widehat{f}_s(\lambda)|^2 ds.$$

Then,

$$\begin{aligned}|\widehat{F}(\lambda)|^2 &\le \|\varphi\|_2^2 \widehat{g}(\lambda) &(2.3)\\ &= \|\varphi\|_2^2 \sum_{n\in\mathbb{Z}} \int_{\mathbb{T}}\int_{\mathbb{R}^2} |\widehat{\varphi}^h(x,y,\lambda,z,n)|^2 dx dy dz.\end{aligned}$$

Hence,

$$e^{2b\pi\lambda^2}|\widehat{F}(\lambda)|^2 \leq \|\varphi\|_2^2 e^{2b\pi\lambda^2}\sum_{n\in\mathbb{Z}}\int_{\mathbb{T}}|\lambda|\|(\sigma_{\lambda,z}\otimes\chi_n)(f)\|_{\mathrm{HS}}^2 dz$$
$$\leq \|\varphi\|_2^2 e^{2b\pi\lambda^2}C_1C^2\sum_{n\in\mathbb{Z}}\int_{\mathbb{T}}\frac{|\lambda|\|(\sigma_{\lambda,z}\otimes\chi_n)(f)\|_{\mathrm{HS}}^2 e^{2b\pi|n|^2}}{C^2}\left(\frac{e^{-2b\pi|n|^2}}{C_1}\right)dz$$

where C_1 is chosen so that $\frac{e^{-2b\pi|n|^2}}{C_1}$ is the density of a discrete probability measure. Let

$$J = \frac{1}{\|\varphi\|_2^2 C_1 C^2}e^{2b\pi\lambda^2}|\widehat{F}(\lambda)|^2\log^+\left(\frac{e^{b\pi\lambda^2}|\widehat{F}(\lambda)|}{C\sqrt{\|\varphi\|_2^2 C_1}}\right)$$

and

$$A = e^{b\pi(\lambda^2+|n|^2)}\left(\int_{\mathbb{R}^2}|\widehat{\varphi}^h(x,y,\lambda,z,n)|^2dxdy\right)^{\frac{1}{2}}.$$

Using inequalities (1.2) and (1.1), we get:

$$J \leq 2\sum_{n\in\mathbb{Z}}\int_{\mathbb{T}}\frac{A^2}{C^2}\log^+\left(\frac{A}{C}\right)\frac{e^{-2b\pi|n|^2}}{C_1}dz.$$

This proves that:

$$\int_{\mathbb{R}}e^{2b\pi\lambda^2}|\widehat{F}(\lambda)|^2\log^+\left(\frac{e^{b\pi\lambda^2}|\widehat{F}(\lambda)|}{C\sqrt{\|\varphi\|_2^2C_1}}\right)d\lambda$$
$$\leq 2\|\varphi\|_2^2\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\int_{\mathbb{T}}A^2\log^+\left(\frac{A}{C}\right)dzd\lambda$$
$$< +\infty,$$

as was to be shown. □

Using Corollary 1.1, we get that $F = 0$ almost everywhere on $\mathbb{R}$ if $ab > 1$. Then $f = 0$ almost everywhere on G. When $ab = 1$, it holds that $F(t) = c_\varphi e^{-a\pi t^2}(t\in\mathbb{R})$ for some constant c_φ depending upon φ and then for all $t\in\mathbb{R}$ and $s\in\Sigma$, it follows that $f(t,s) = e^{-a\pi t^2}f(0,s)$.

Let now χ be a character of $\mathbb{T}$ and g_3 a measurable function on $\mathbb{T}$. We consider the function g_3^χ on $\mathbb{T}$ defined by $g_3^\chi(x) = \int_{\mathbb{T}}g_3(xk)\chi(k)dk$. Then $\chi_n(g_3^\chi) = \chi_n(g)\int_{\mathbb{T}}\overline{\chi(k)}\chi_n(k)dk$ and hence $\chi_n(g_3^\chi) = 0$ whenever χ_n is not a multiple of χ by Schur orthogonality relations.

Assume now $ab < 1$, for any r with $a < r < \frac{1}{b}$. Let

$$f_r(t,z,m,u) = \varphi_r(t,z)g_2(m)g_3^\chi(u),$$

where $\varphi_r(t,z)$ the function φ_r defined by:

$$\varphi_r(t,z) = e^{-r\pi t^2}\varphi(z), \tag{2.4}$$

where $\varphi\in C_c(\mathbb{C})$ and g_2 is a function on $\mathbb{Z}$ of finite support.

This means that there is $M' > 0$ such that

$$\sum_{n\in\mathbb{Z}}e^{2b\pi|n|^2}\ |\chi_n(g_3^\chi)|^2 < M'.$$

We obtain for $\lambda\in\mathbb{R}^\star$, $z\in\mathbb{T}$ and $n\in\mathbb{Z}$ that

$$(\sigma_{\lambda,z}\otimes\chi_n)(f_r) = \pi_\lambda(\varphi_r)\ \chi_n(g_3^\chi)\ \sum_{m\in\mathbb{Z}}e^{-imz}g_2(m)$$

and therefore

$$\|(\sigma_{\lambda,z}\otimes\chi_n)(f_r)\|_{\mathrm{HS}}$$
$$= \|\pi_\lambda(\varphi_r)\|_{\mathrm{HS}}\ |\chi_n(g_3^\chi)|\ |\sum_{m\in\mathbb{Z}}e^{-imz}g_2(m)|$$

Let

$$A' = \|(\sigma_{\lambda,z}\otimes\chi_n)(f_r)\|_{\mathrm{HS}}$$

and

$$I = \int_{\mathbb{R}}\int_{\mathbb{T}}\sum_{n\in\mathbb{Z}}e^{2b\pi(\lambda^2+|n|^2)}|\lambda|A'^2\log^+\left(e^{2b\pi(\lambda^2+|n|^2)}|\lambda|A'^2\right)dzd\lambda.$$

Then using Miyachi's Theorem for Heisenberg Lie groups, we get:

$$\begin{aligned} I &\leq \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\int_{\mathbb{T}} e^{2b\pi(\lambda^2+|n|^2)}|\lambda|\|\pi_\lambda(\varphi_r)\|^2_{\mathrm{HS}}\ |\chi_n(g_3^\chi)|^2\ |\sum_{m\in\mathbb{Z}} e^{-imz}g_2(m)|^2 \\ &\quad\times \log^+\left(e^{2b\pi(\lambda^2+|n|^2)}|\lambda|\|\pi_\lambda(\varphi_r)\|^2_{\mathrm{HS}}\ |\chi_n(g_3^\chi)|^2\ |\sum_{m\in\mathbb{Z}} e^{-imz}g_2(m)|^2\right) dzd\lambda, \\ &\leq M'\int_{\mathbb{T}}|\sum_{m\in\mathbb{Z}} e^{-imz}g_2(m)|^2dz\int_{\mathbb{R}} e^{2b\pi\lambda^2}|\lambda|\|\pi_\lambda(\varphi_r)\|^2_{\mathrm{HS}}\log^+\left(\frac{e^{2b\pi\lambda^2}|\lambda|\|\pi_\lambda(\varphi_r)\|^2_{\mathrm{HS}}}{M}\right)d\lambda. \\ &\leq M''\int_{\mathbb{R}}\frac{1}{r}e^{2\pi\lambda^2(b-\frac{1}{r})}\|\varphi\|^2_{L^2(\mathbb{C})}\log^+\left(\frac{e^{2b\pi\lambda^2}|\lambda|\|\pi_\lambda(\varphi_r)\|^2_{\mathrm{HS}}}{M}\right)d\lambda \leq M_1\int_{\mathbb{R}}\log^+\left(\frac{e^{2b\pi\lambda^2}|\lambda|\|\pi_\lambda(\varphi_r)\|^2_{\mathrm{HS}}}{M}\right)d\lambda, \end{aligned}$$

which is finite. Here, M', M'' and M_1 are positive constants, and this completes the proof of our main Theorem 2.1. □

In a final section, we derive some consequences of the last theorem.

3 Some consequences

3.1 A refined analogue of Hardy's Theorem

As mentioned above, A. Baklouti and E. Kaniuth (cf. [6]) produced an analogue of Hardy's Theorem for oscillator groups. More precisely, they proved the following result.

Theorem 3.1 (A. Baklouti and E. Kaniuth, 2010)**.** *Let G be an oscillator groups, $a, b \in \mathbb{R}^*_+$. Let $f : G \to \mathbb{C}$ be a measurable function on G satisfying the following decay conditions:*

(i) $|f(t,s)| \leq e^{-a\pi t^2}\varphi(s)$, for all $t \in \mathbb{R}$ and $s \in \Sigma$ where $\varphi \in L^1(\Sigma)\cap L^2(\Sigma)$,

(ii) $\|(\sigma_{\lambda,z}\otimes\chi_n)(f)\|_{\mathrm{HS}} \leq e^{-b\pi\lambda^2}\psi(n)$, for all $\lambda \in \mathbb{R}, \lambda \neq 0, z \in \mathbb{T}$ and $n \in \mathbb{Z}$ where $\psi \in l^2(\mathbb{Z})$.

If $ab > 1$, then $f = 0$ almost everywhere. If $ab = 1$, then there are many functions of the form $f(t,s) = e^{-a\pi t^2}g(s)$, $g \in C_c(\Sigma)$, satisfying (i) and (ii). If $ab < 1$, then there are infinitely many linearly independent functions satisfying (i) and (ii).

We first state a refined version of Theorem 3.1.

Corollary 3.2. *Let G be an oscillator groups, $a, b \in \mathbb{R}^*_+$. Let $f : G \to \mathbb{C}$ be a measurable function on G satisfying the following decay conditions:*

(i) $|f(t,s)| \leq ce^{-a\pi\|t,s\|^2}$, for all $t \in \mathbb{R}$ and $s \in \Sigma$,

(ii) $\left(\int_{\mathbb{R}^2}|\widehat{\varphi}^h(x,y,\lambda,z,n)|^2dxdy\right)^{\frac{1}{2}} \leq ce^{-b\pi(\lambda^2+|n|^2)}$, for all $\lambda \in \mathbb{R}, \lambda \neq 0, z \in \mathbb{T}$ and $n \in \mathbb{Z}$ where $c \in \mathbb{R}$.

If $ab > 1$, then $f = 0$ almost everywhere. If $ab = 1$, then $f(t,s) = e^{-a\pi t^2}f(0,s)$ for all $t \in \mathbb{R}$ and $s \in \Sigma$. If $ab < 1$, then there are infinitely many linearly independent functions satisfying (i) and (ii).

As a second consequence of this Theorem. We prove the following result :

Theorem 3.3 (The Cowling-Price Theorem for oscillator groups)**.** *Let G be an oscillator groups, $a, b \in \mathbb{R}^*_+$. Let $f : G \to \mathbb{C}$ be a measurable function on G satisfying the following decay conditions:*

(i) $I_a(f) = \int_G e^{ap\pi\|(t,s)\|^2}|f(t,s)|^pdtds < +\infty$,

(ii) $J_b(f) = \int_{\mathbb{R}}\int_{\mathbb{T}}\sum_{n\in\mathbb{Z}} e^{qb\pi(\lambda^2+|n|^2)}\left(\int_{\mathbb{R}^2}|\widehat{\varphi}^h(x,y,\lambda,z,n)|^2dxdy\right)^{\frac{q}{2}}dzd\lambda < +\infty$,

with $2 \leq p, q \leq +\infty$ and $\min(p,q) < +\infty$. If $ab \geq 1$, then $f = 0$ almost everywhere . If $ab < 1$, then there are infinitely many linearly independent functions satisfying (i) and (ii).

Proof. The function $e^{a\|\cdot\|^2}f$ belongs to $L^p(G)$ by assumption, and hence to $L^2(G) + L^\infty(G)$. On the other hand, whenever $q > 2$, let

$$A = e^{b\pi(\lambda^2+|n|^2)}\left(\int_{\mathbb{R}^2}|\widehat{\varphi}^h(x,y,\lambda,z,n)|^2dxdy\right)^{\frac{1}{2}}$$

, we get that $(q-2)\log^+ A \leq A^{q-2}$ and then

$$(q-2)A^2\log^+(A) \leq$$

$$e^{qb\pi(\lambda^2+|n|^2)}\left(\int_{\mathbb{R}^2}|\widehat{\varphi}^h(x,y,\lambda,z,n)|^2dxdy\right)^{\frac{q}{2}}.$$

This leads to

$$\int_{\mathbb{R}}\int_{\mathbb{T}}\sum_{n\in\mathbb{Z}} A^2\log^+(A)dzd\lambda$$

$$\leq \frac{1}{(q-2)}\int_{\mathbb{R}}\int_{\mathbb{T}}\sum_{n\in\mathbb{Z}} A^q dzd\lambda < +\infty.$$

Hence, the function f fulfills the second decay condition of Theorem 2.1. This implies that f vanishes almost everywhere whenever $ab > 1$.

In the case where $ab = 1$, it holds that $f(t,s) = f(0,s)e^{-a\pi t^2}$ for all $t \in \mathbb{R}$ and $s \in \Sigma$. This entails that the integral

$$\int_G e^{pa\pi\|x\|^2}|f(x)|^p dx$$

diverges unless $f(0,\cdot)$ vanishes almost everywhere on Σ, as was to be shown.

We now look at the case where $q = 2$. The function $e^{a\pi|\cdot|^2}F$ belongs to $L^p(\mathbb{R})$, where F is the function defined in the proof of Theorem 2.1 and $e^{b\pi|\cdot|^2}\widehat{F}$ belongs to $L^2(\mathbb{R})$. In fact, by using inequality (1.1) and Hölder's inequality, we get:

$$\begin{aligned}
&\int_{\mathbb{R}} |F(t)|^p e^{ap\pi t^2} dt \\
&\leq \int_{\mathbb{R}} \left(\int_{\Sigma} |f(t,s)||\varphi(s)| e^{a\pi t^2} ds \right)^p dt \\
&\leq \left(\int_{\Sigma} \left(\int_{\mathbb{R}} |f(t,s)|^p |\varphi(s)|^p e^{pa\pi t^2} dt \right)^{1/p} ds \right)^p \\
&\leq \left(\int_{\Sigma} e^{-a\pi\|s\|^2} \left(\int_{\mathbb{R}} |f(t,s)|^p |\varphi(s)|^p e^{pa\pi t^2} e^{a\pi p\|s\|^2} dt \right)^{1/p} ds \right)^p \\
&\leq M \int_{\Sigma}\int_{\mathbb{R}} |f(t,s)|^p e^{a\pi p\|(t,s)\|^2} dt ds < +\infty,
\end{aligned}$$

for some positive constant M. Now, using inequality (2.3) we get:

$$\begin{aligned}
&\int_{\mathbb{R}} e^{2b\pi\lambda^2} |\widehat{F}(\lambda)|^2 d\lambda \leq \ E' \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} e^{2b\pi\lambda^2} |\lambda| \|(\sigma_{\lambda,z}\otimes\chi_n)(f)\|^2_{\mathrm{HS}} dz d\lambda \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} e^{qb\pi(\lambda^2+|n|^2)} |\lambda|^{\frac{dq}{2}} \|(\sigma_{\lambda,z}\otimes\chi_n)(f)\|^q_{\mathrm{HS}} dz d\lambda < +\infty.
\end{aligned}$$

By the classical Cowling-Price Theorem for the real line, F is zero almost everywhere whenever $ab \geq 1$ and hence so is f. If $ab < 1$, for any r with $a < r < \frac{1}{b}$, let f_r be defined on G as in the proof of Theorem 2.1. We obtain that f_r verifies (i) and (ii).

In fact,

$$\begin{aligned}
& I_a(f_r) \\
=& \int_{\mathbb{R}}\int_{\mathbb{C}}\int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} e^{ap\pi t^2} e^{ap\pi|n|^2} e^{ap\pi|z|^2} |\varphi_r(t,z)|^p |g_2(n)|^p |g_3(u)|^p du dz dt \\
=& \int_{\mathbb{R}}\int_{\mathbb{C}}\int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} e^{(a-r)p\pi t^2} e^{ap\pi|n|^2} e^{ap\pi|z|^2} |\varphi(z)|^p |g_2(n)|^p |g_3(u)|^p du dz dt < +\infty.
\end{aligned}$$

On the other hand,

$$\|(\sigma_{\lambda,z}\otimes\chi_n)(f_r)\|_{\mathrm{HS}} = \|\pi_\lambda(\varphi_r)\|_{\mathrm{HS}} \ |\chi_n(g_3^\chi)| \ \Big|\sum_{m\in\mathbb{Z}} e^{-imz} g_2(m)\Big|.$$

We get then

$$\begin{aligned}
J_b(f_r) &= \int_{\mathbb{R}}\int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} e^{qb\pi(\lambda^2+|n|^2)} |\lambda|^{\frac{dq}{2}} \|\pi_\lambda(\varphi_r)\|^q_{\mathrm{HS}} \ |\chi_n(g_3^\chi)|^q \ \Big|\sum_{m\in\mathbb{Z}} e^{-imz} g_2(m)\Big|^q \\
&= \sum_{n\in\mathbb{Z}} e^{qb\pi|n|^2} \ |\chi_n(g_3^\chi)|^q \int_{\mathbb{T}} \Big|\sum_{m\in\mathbb{Z}} e^{-imz} g_2(m)\Big|^q dz \int_{\mathbb{R}} e^{qb\pi(\lambda^2)} \|\pi_\lambda(\varphi_r)\|^q_{\mathrm{HS}} |\lambda|^{\frac{dq}{2}} d\lambda \\
&\leq K \frac{1}{r^{\frac{q}{2}}} \|\varphi\|^q_{L^2(\mathbb{C})} \int_{\mathbb{R}} e^{q\pi\lambda^2(b-\frac{1}{r})} d\lambda < +\infty
\end{aligned}$$

for some constant K. □

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