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On the fourth-order Joseph-Lundgren exponent

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Abstract

In this paper, we revise the proof of the existence and the explicit value of the fourth-order Joseph–Lundgren exponent $p_c(n,4)$ computed by Gazzola and Grunau [9](see also [10, 5]). Inspired by our work in [13], we propose a simple proof based on a symmetry property that allows us to reduce the computation of $p_c(n,4)$ into solving a second degree polynomial equation. Therefore we can easily derive the explicit value of this exponent. Our approach is much more streamlined and more transparent compared to [9, 10] in terms of finding this explicit value.

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1 Introduction

In the last decades, problems related to the nonexistence of finite Morse index sign-changing solutions for second- order and fourth-order Lane–Emden equations on unbounded domains of $\mathbb{R}^n$ have received a lot of attention (see [2, 4, 8, 21, 22, 3, 12, 5, 14, 15, 16]).

We first recall that the corresponding second order equation

$$-\Delta u = |u|^{p-1}u, \qquad \text{in } \mathbb{R}^n,\ p>1, \tag{1.1}$$

was explored in the subcritical case by Bahri and Lions using Moser iteration techniques combined with the well known Pohozaeav identity [2]. These results was extended in the supercritical case by Dancer [4] with the restriction $\frac{n+2}{n-2} < p < \frac{n}{n-4}$, if $n \geq 4$. In an elegant paper, Farina [8] proved that nontrivial finite Morse index solutions to (1.1) exist if and only if $p \geq p_c(n,2)$ and $n \geq 11$, or $p = \frac{n+2}{n-2}$ and $n \geq 3$, where $p_c(n,2)$ is the so-called Joseph-Lundgren exponent. His proof makes a delicate use of the classical Moser iteration method: namely multiply the equation (1.1) by the power of u, like u^q; $q>1$, Moser's iteration works because of the following simple identity.

$$\int_{\mathbb{R}^n} u^q(-\Delta u)dx = \frac{4q}{(q+1)^2}\int_{\mathbb{R}^n} |\nabla u^{\frac{q+1}{2}}|^2 dx,$$

$\forall\ u \in C_0^1(\mathbb{R}^n)$. Motivated by [8, 20], sharp classification results are obtained in strips [6], which seem as the nonexistence and the existence results in a strictly star-shaped bounded domain derived from Pohozaev identity.

Regarding for fourth order equation

$$\Delta^2 u = |u|^{p-1}u,\ \text{in } \mathbb{R}^n,\ p>1, \tag{1.2}$$

Ramos and Rodriguez obtained classification for finite Morse index sign-changing solutions in the subcritical case (see [18]). The supercritical case is more complicated and there are several new approaches dealing with (1.2). The first device consists to use $v = -\Delta u$ as a test function combined with the following Souplet's inequality [19]: for $u>0$ satisfying (1.2) it holds

$$-\Delta u \geq \sqrt{\frac{2}{p+1}}\,u^{\frac{p+1}{2}}.$$

Therefore, the exponent $\frac{n}{n-8} + \epsilon_n$ for some $\epsilon_n > 0$ has been discovered in [21]. These results were improved in [22] by adapting Farina's approach with the restriction on the power $q < \frac{2}{3}$. The second approach obtained by Cowan and Ghoussoub [3], Dupaigne, Ghergu, Goubet and Warnault [7] and further exploited by Hajlaoui, Ye and one of the authors [12] in order to derive the following interesting interpolated version of the inequality: for stable positive solutions to (1.2), it holds

$$\sqrt{p}\int_{\mathbb{R}^n} u^{\frac{p-1}{2}}\psi^2 dx \leq \int_{\mathbb{R}^n} |\nabla\psi|^2 dx,\ \ \forall\ \psi \in C_0^1(\mathbb{R}^n).$$

This approach improves the first upper bound $\frac{n}{n-8}+\epsilon_n$, but it again fails to catch the optimal exponent $p_c(4,n)$. It should be remarked that by combining these two approaches one can show that stable positive solutions to (1.2) do not exist when $n \leq 12$ and $p > \frac{n+4}{n-4}$, see [12]. Recently, Dávila, Dupaigne, Wang and Wei [5] have derived a monotonicity formula for solutions of (1.2) to reduce the nonexistence of nontrivial entire solutions for the problem (1.2), to that of nontrivial homogeneous solutions, and gave a complete classification of stable solutions and those of finite Morse index solutions.

In the present paper we investigate the existence and then derive the explicit value of the fourth-order

Joseph-Lundgren exponent for the equation (1.2), where $p > \frac{n+4}{n-4}$ and $n > 4$. Let us mention that the fourth-order Joseph-Lundgren exponent was computed by Gazzola and Grunau [9](see also [10, 5]). There are several motivations for the study of (1.2). Let us try to explain them in some detail. Most of the methods employed for the proof of proposition 2 in [9] are complicated and difficult. For instance, qualitative properties of solutions require a detailed analysis of a dynamical system in the corresponding phase space which is four dimensional for (1.2).

In radial coordinates $r = |x|$, equation (1.2) reads

$$u^{(4)}(r) + \frac{2(n-1)}{r}u^{(3)}(r) + \frac{(n-1)(n-3)}{r^2}u''(r) - \frac{(n-1)(n-3)}{r^3}u'(r) = u^p(r), \quad r \in [0,\ \infty). \quad (1.3)$$

In [9], Gazzola and Grunau transformed (1.3) first into an autonomous equation and, subsequently, into an autonomous system. For some of the estimates that will follow, it is convenient to rewrite the original assumption $p > \frac{n+4}{n-4}$ as

$$(n-4)(p-1) > 8.$$

Inspired by the proof of ([20], Proposition 3.7) (see also [11, 17]), they used

$$u(r) = r^{\frac{-4}{p-1}}v(\log r) \ \ (r > 0), \quad v(t) = e^{\frac{4t}{p-1}}u(e^t) \quad (1.4)$$

for $t \in \mathbb{R}$. Therefore, after the change (1.4), equation (1.3) may be rewritten as

$$v^{(4)}(t) + K_3 v^{(3)}(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = v^p(t), \quad t \in \mathbb{R}, \quad (1.5)$$

where the constants K_i, $i = 0, ..., 3$ are defined in ([9] page 911). Note that (1.5) admits the two constant solutions $v_0 \equiv 0$ and $v_s \equiv K_0^{\frac{1}{p-1}}$ which, by (1.4), correspond to the following solutions of (1.3):

$$u_0(r) = 0, \quad u_s(r) = K_0^{\frac{1}{p-1}} r^{\frac{-4}{p-1}}.$$

They wrote (1.5) as a system in $\mathbb{R}^4$. This system has the two stationary points (corresponding to v_0 and v_s)

$$O(0,0,0,0) \ \text{ and } \ P(K_0^{\frac{1}{p-1}}, -\frac{4}{p-1}K_0^{\frac{1}{p-1}}, 0, 0).$$

The “regular point” O is a hyperbolic point, while the stable and the unstable manifolds are two-dimensional. The “singular point” P is described by:

> For any $n \geq 13$ there exists $p_c(n,4) > \frac{n+4}{n-4}$ such that for any $p \geq p_c(n,4)$, P is stable.

The number $p_c(n,4)$ is the unique value of $p > \frac{n+4}{n-4}$ such that

$$\begin{aligned} &-(n-4)(n^3-4n^2-128n+256)(p-1)^4 \\ &+128(3n-8)(n-6)(p-1)^3 \\ &+256(n^2-18n+52)(p-1)^2 \\ &-2048(n-6)(p-1)+4096 \\ &=0. \end{aligned} \quad (1.6)$$

The purpose of this paper is to first provide an implicit existence of the fourth-order Joseph-Lundgren exponent $p_c(n,4)$ in the supercritical range. Furthermore, our approach enables us to find an explicit expression of $p_c(n,4)$ depending on the dimension n.

First, let us recall the Hardy-Rellich inequality and its consequences.

Theorem 1.1. *[1] Let $\psi \in H^2(\mathbb{R}^n)$ and $n \geq 4$. Then*

$$\int_{\mathbb{R}^n} |\Delta\psi|^2 dx \geq \frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n} \frac{\psi^2}{|x|^4}dx.$$

Now set

$$\alpha = \frac{4}{p-1}, \quad K_0 = \alpha(\alpha+2)(\alpha-n+4)(\alpha-n+2),$$

and

$$\begin{aligned} Q(\alpha) : &= pK_0 - \frac{n^2(n-4)^2}{16} \\ &= (\alpha+2)(\alpha+4)(\alpha-n+2)(\alpha-n+4) - \frac{n^2(n-4)^2}{16}. \end{aligned}$$

Then

$$u_s(r) = K_0^{\frac{1}{p-1}} r^{\frac{-4}{p-1}}$$

is a singular solution to (1.2) in $\mathbb{R}^n \backslash \{0\}$. In view of theorem 1.1, u_s is stable if and only if

$$pK_0 \leq \frac{n^2(n-4)^2}{16}, \quad (1.7)$$

that is

$$Q(\alpha) \leq 0.$$

In order to get a better range of the power p from (1.7), it is necessary for us to study the following equation

$$Q(\alpha) = 0. \quad (1.8)$$

Remarks 1.1.

(i) We can only consider the behavior of (1.6) for $p > \frac{n+4}{n-4}$. Through tedious computations, we see the equation which appeared in (1.8) is the simplified form of (1.6).

(ii) As for second order equations, our analysis reveals the existence of a new critical exponent $p_c(n,4)$ (larger than the classical critical exponent $\frac{n+4}{n-4}$) such that the singular solution u_s of (1.2) is stable if $p \geq p_c(n,4)$. In particular, we give an explicit expression of $p_c(n,4)$ depending on the dimension n.

Our paper is essentially focused on the existence of a new critical exponent and stability properties of singular solution u_s of (1.2). The following Proposition is crucial for our approach and exhibits a fourth-order Joseph-Lundgren exponent $p_c(n,4) > \frac{n+4}{n-4}$ such that for $p > \frac{n+4}{n-4}$, (1.7) holds true if and only if $p \geq p_c(n,4)$.

Proposition 1.1. *Let $n > 4$ and $p > \frac{n+4}{n-4}$.*

(1) • If $n \leq 12$, then condition (1.7) does not hold for any $p > \frac{n+4}{n-4}$.

• If $n \geq 13$, then the equation

$$Q(\alpha) := pK_0 - \frac{n^2(n-4)^2}{16} = 0$$

has a unique solution $p_c(n,4) > \frac{n+4}{n-4}$, and condition (1.7) holds if and only if $p \geq p_c(n,4)$.

(2) Moreover

$$p_c(n,4) = \begin{cases} +\infty & \text{if } n \leq 12, \\ \dfrac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \geq 13. \end{cases}$$

The proof of Proposition 1.1 is less complicated and more transparent compared to [9] in which the implicit existence of $p_c(n,4)$ was proved. In fact, the polynomial Q is of degree four, but when compared to [9], it has a complicated expression. To overcome this difficulty, we make use of the symmetry $Q(\alpha) = Q(n-6-\alpha)$. This allows us to write Q as a second degree polynomial in the variable $\beta = (\alpha - \frac{n-6}{2})^2$. Therefore, we may easily derive the explicit value of $p_c(n,4)$. Clearly, our approach is more transparent than the proof of proposition 2 in [9] in terms of finding the explicit value of $p_c(n,4)$. Also, the interest of our approach is to simplify complicated proofs for future generations since it ensures a clearer visibility and provides a more accessible literature and that the beauty of mathematical sciences is in simplicity.

2 Proof of Proposition 1.1

This section is devoted to the proof of Proposition 1.1.

Proof of Part (1): let

$$\begin{aligned} Q(\alpha) &= pK_0 - \frac{n^2(n-4)^2}{16} \\ &= (\alpha+2)(\alpha+4)(\alpha-n+2)(\alpha-n+4) \\ &\quad -\frac{n^2(n-4)^2}{16}, \end{aligned}$$

where $\alpha = \frac{4}{p-1}$.
Note that

$$p > \frac{n+4}{n-4} \Leftrightarrow 0 < \alpha < \frac{n-4}{2}.$$

Also, one has:

$$\text{for all } \frac{n-8}{2} \leq \alpha < \frac{n-4}{2}, \quad Q(\alpha) > 0. \qquad (2.1)$$

Indeed

$$(\alpha+2)(\alpha+4) \geq \frac{n(n-4)}{4}$$

and

$$(\alpha-n+2)(\alpha-n+4) > \frac{n(n-4)}{4}.$$

Taking in account that

$$\frac{n^2(n-4)^2}{16} - \frac{n^2(n-4)^2}{16} = 0.$$

Then (2.1) follows.
We have also

$$\begin{aligned} \lim_{\alpha\to 0} Q(\alpha) &= \frac{n-4}{16}\left(-n^3+4n^2+128n-256\right) \\ &= L_n\,. \end{aligned}$$

By a direct calculations shows that $L_n > 0$, if $n = 5, 6, ..., 12$ and $L_n < 0$, if $n = 13, 14$. Furthermore, we have for $n \geq 15$;

$$\begin{aligned} -n^3+4n^2+128n-256 &\leq -15n^2+4n^2+128n \\ &\leq -11\times 15n+128n \\ &< 0. \end{aligned}$$

In conclusion, if $n \geq 13$, $L_n < 0$ and $L_n > 0$ if $5 \leq n \leq 12$. Moreover, if $n = 5, ..., 8$, one has

$$\left(0,\ \frac{n-4}{2}\right) \subset \left(\frac{n-8}{2},\ \frac{n-4}{2}\right),$$

then from (2.1), $Q(\alpha) > 0$ for all $0 < \alpha < \frac{n-4}{2}$ which end the proof of proposition 1.1, if $n = 5, ..., 8$.

On the other hand, it is not difficult to see the following symmetry property of Q:

$$Q(\alpha) = Q(n-6-\alpha),$$

which implies that $Q'(\frac{n-6}{2}) = Q^{(3)}(\frac{n-6}{2}) = 0$. Hence, by Taylor's formula we derive

$$\begin{aligned} Q(\alpha) = \quad & \left(\alpha - \frac{n-6}{2}\right)^4 - \frac{n^2+4}{2}\left(\alpha - \frac{n-6}{2}\right)^2 \\ & + \frac{(n-2)^2(n+2)^2 - n^2(n-4)^2}{16}. \end{aligned}$$

If we set $\beta = (\alpha - \frac{n-6}{2})^2$ and

$$A(\beta) = \beta^2 - \frac{n^2+4}{2}\beta + \frac{(n-2)^2(n+2)^2 - n^2(n-4)^2}{16},$$

we therefore get that $Q(\alpha) = A(\beta)$.

It is not hard to show that A has the following properties:

Lemma 2.1.
(1) $A'(\beta) = 2\beta - \frac{n^2+4}{2} = 2(\beta - \beta_0)$, *where* $\beta_0 = \frac{n^2+4}{4}$.
(2) If $n \geq 9$, *then* A *is a decreasing function on* $\left(1,\ \left(\frac{n-6}{2}\right)^2\right)$.
(3) $\beta : \left(0,\ \frac{n-8}{2}\right) \to \left(1,\ (\frac{n-6}{2})^2\right)$ *is a bijection map.*

Proof of Lemma 2.1. Part (1) follows easily. For Part (3), one has $\beta\left(\left(0,\ \frac{n-8}{2}\right)\right) = \left(1,\ (\frac{n-6}{2})^2\right)$ and $\alpha - \frac{n-6}{2} \in \left(-\frac{n-6}{2},\ -1\right) \subset (-\infty,\ \mathbf{0})$, for every $\alpha \in \left(0,\ \frac{n-8}{2}\right)$. Thus, the equation $\beta(\alpha) = y$, with $y \in \left(1,\ (\frac{n-6}{2})^2\right)$, has a unique solution $\alpha = -\sqrt{y} + \frac{n-6}{2} \in \left(0,\ \frac{n-8}{2}\right)$. In fact $\alpha = \sqrt{y} + \frac{n-6}{2}$ is rejected because $\alpha - \frac{n-6}{2} = \sqrt{y} > 0$.

It remains to prove Part (2). In view of Part (1), A is a decreasing function on $(-\infty, \beta_0)$. The proof of Part (2) will be complete, if we verify that

$$\left(\frac{n-6}{2}\right)^2 < \beta_0, \text{ if } n \geq 9.$$

Indeed, since $12n - 32 \geq 76$ if $n \geq 9$, then

$$\beta_0 - \left(\frac{n-6}{2}\right)^2 = \frac{12n-32}{4} \geq \frac{76}{4} > 0.$$

This implies that

$$\left(\frac{n-6}{2}\right)^2 < \beta_0 \text{ if } n \geq 9.$$

and completes the proof of the Lemma. □

We are now able to complete the proof of Part (1) of Proposition 1.1. Two cases occur.

Case 1. $n \geq 13$
Recall that $Q(\alpha) > 0$ for all $\alpha \in \left(\frac{n-8}{2},\ \frac{n-4}{2}\right)$,

$$Q\left(\frac{n-8}{2}\right) > 0$$

and $L_n < 0$. Then, Q has at least one root in $\left(0,\ \frac{n-8}{2}\right)$ noted α_c which implies that A has at least one root in $\left(1,\ (\frac{n-6}{2})^2\right)$ noted β_c. In view of Part (2) of Lemma 2.1, we may deduce that β_c is unique if $n \geq 9$. Now, from Part (3) of Lemma 2.1, we also derive that α_c is unique and $\alpha_c = -\sqrt{\beta_c} + \frac{n-6}{2} \in \left(0,\ \frac{n-8}{2}\right)$. In conclusion, Q has a unique root α_c in $\left(0,\ \frac{n-4}{2}\right)$, since $Q(\alpha) > 0$ for all $\alpha \in \left(\frac{n-8}{2},\ \frac{n-4}{2}\right)$. Moreover, one has

$$Q(\alpha) < 0, \text{ on } (0,\ \alpha_c)\ , \text{ and} \qquad (2.2)$$
$$Q(\alpha) > 0, \text{ on } \left(\alpha_c,\ \frac{n-4}{2}\right).$$

Case 2. $9 \leq n \leq 12$.
Using again $Q(\alpha) > 0$ for all $\frac{n-8}{2} < \alpha < \frac{n-4}{2}$. By Lemma 2.1, we have A is a decreasing function on $\left(1,\ \left(\frac{n-6}{2}\right)^2\right)$ and

$$\lim_{\beta \to \left(\frac{n-6}{2}\right)^2} A(\beta) = \lim_{\alpha \to 0} Q(\alpha) = L_n > 0,$$

then $A(\beta) > 0$ for all $1 < \beta < \left(\frac{n-6}{2}\right)^2$. This implies that $Q(\alpha) > 0$ for all $0 < \alpha < \frac{n-8}{2}$. As a consequence, if $5 \leq n \leq 12$,

$$Q(\alpha) > 0 \ \ \forall \ \ 0 < \alpha < \frac{n-4}{2}. \qquad (2.3)$$

Finally, note that

$$\alpha_c = \frac{4}{p_c(n,4) - 1} \text{ and } \alpha_c = -\sqrt{\beta_c} + \frac{n-6}{2}.$$

Then

$$p_c(n,4) = \frac{n+2-2\sqrt{\beta_c}}{n-6-2\sqrt{\beta_c}}. \qquad (2.4)$$

On the other hand, by (2.3) and (2.2), we have
- If $n \leq 12$, then condition (1.7) does not hold for any $p > \frac{n+4}{n-4}$.
- If $n \geq 13$, the condition (1.7) holds if and only if $p \geq p_c(n,4)$.

Proof of Part (2). For $n \geq 13$, we are now in position to solve the equation

$$A(\beta) = 0, \ \ 1 < \beta < \left(\frac{n-6}{2}\right)^2, \ i.e.$$

$$\beta^2 - \frac{n^2+4}{2}\beta + \frac{(n-2)^2(n+2)^2 - n^2(n-4)^2}{16} = 0, \qquad (2.5)$$
$$1 < \beta < \left(\frac{n-6}{2}\right)^2.$$

The discriminant of (2.5) is

$$\begin{aligned}\Delta &= \frac{n^2}{4}((n-4)^2 + 16)\\ &= \frac{n^2}{4}(n^2 - 8n + 32).\end{aligned}$$

Since $n \geq 13$, then $\Delta > 0$.
This implies that the equation (2.5) has two real solutions

$$\beta_1 = \frac{n^2+4-2\sqrt{\Delta}}{4} \text{ and } \beta_2 = \frac{n^2+4+2\sqrt{\Delta}}{4}.$$

If $n \geq 13$, then $(\frac{n-6}{2})^2 - \beta_2 = \frac{-12n+32-2\sqrt{\Delta}}{4} < 0$ and recall that A has a unique root $\beta_c \in \left(0,\ \left(\frac{n-6}{2}\right)^2\right)$ for any $n \geq 13$. Hence

$$\beta_1 = \beta_c. \qquad (2.6)$$

From (2.4) and (2.6), we deduce that

$$p_c(n,4) = \frac{n+2-2\sqrt{\beta_c}}{n-6-2\sqrt{\beta_c}} = \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}.$$

The latter concludes the proof of Proposition 1.1. □

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